Math 2700 Elementary Differential Equations

1.1 Modeling via differential equations

Consider a quantity which changes w/ time.

The rate of change (derivative) may depend on its current value.

Ex 1 Rat population in Athens $P(t)$

Its rate of change (growth) is proportional to the current population (say, coefficient is $k > 0$)

$$\rightsquigarrow \frac{dP}{dt} = kP$$

$$\frac{d}{dt}(e^{kt}) = e^{kt} \cdot k$$

$$\frac{d}{dt}(2e^{kt}) = 2e^{kt} \cdot k$$

General solution : $P(t) = Ce^{kt}$

(for any constant C)

The function $P(t)$ can be determined if we know its initial value $P(t_0) = P_0$

$$P_0 = Ce^{kt_0} \qquad C = P_0 e^{-kt_0}$$

$\rightarrow$ the unique sol'n satisfying the initial condition $P(t_0) = P_0$ is

$$P(t) = P_0 e^{-kt_0} e^{kt} = P_0 e^{k(t-t_0)}$$ "particular sol'n"

- If $P_0 > 0$, then the population grows exponentially in time.

Ex 2 An improved model: logistic population model.

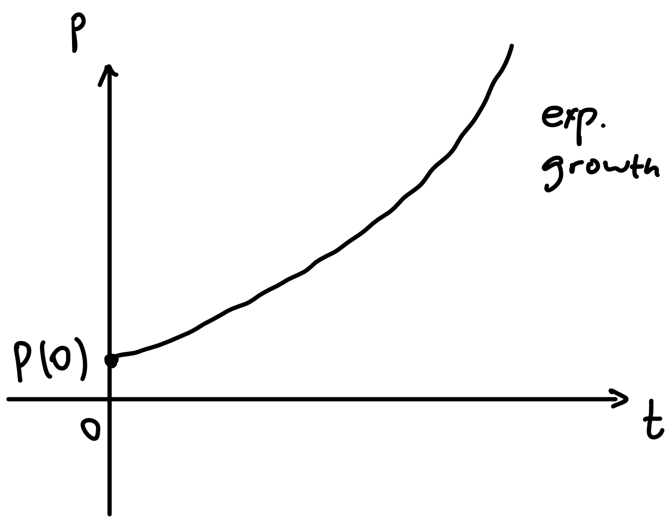
Population growth is restricted by the environment

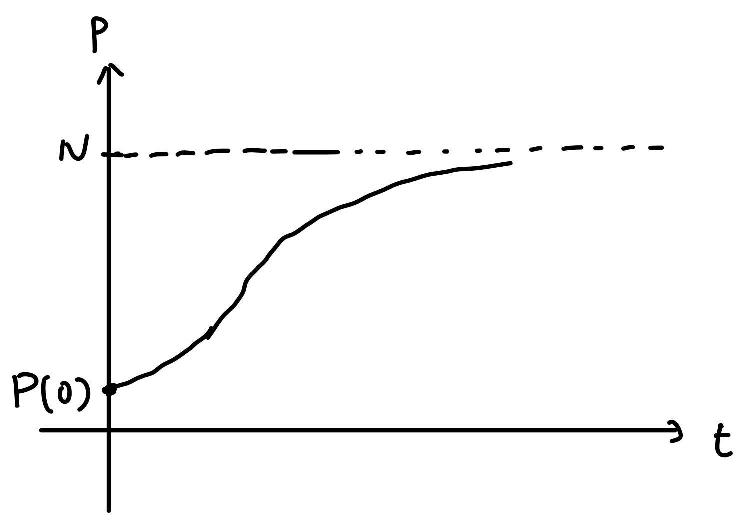
Constant parameter N: carrying capacity of the environment.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

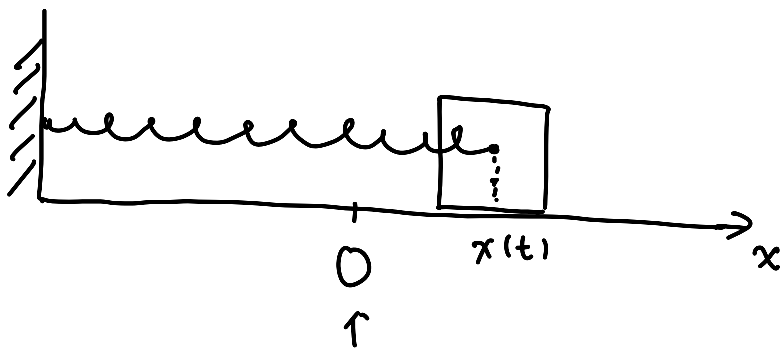
- When $P > N$, then $\frac{dP}{dt} < 0 \rightsquigarrow$ population decreases.
- When P is much less than N, then $\frac{dP}{dt} \approx kP$

(similar to exponential growth)





Ex 3 Harmonic oscillator



$x(t)$: position of the block

m: mass of the block

Hooke's law: magnitude of the force from the spring is $k|x|$ where $k>0$ is the spring constant.

Newton's 2nd law:

$$m \frac{d^2x}{dt^2} = -kx$$

"a second order differential equation"

Ex 4 A chemical reaction

$$A + 2B \longrightarrow 4D$$

$c_A(t)$, $c_B(t)$, $c_D(t)$: concentration of chemicals

The reaction rate is proportional to $c_A c_B^2$ (say, coeff is $k>0$)

$$\begin{cases} \dfrac{dc_A}{dt} = -k c_A c_B^2 \\[2mm] \dfrac{dc_B}{dt} = -2k c_A c_B^2 \\[2mm] \dfrac{dc_D}{dt} = 4k c_A c_B^2 \end{cases}$$

"a system of differential equations"

- Generally, an (ordinary) differential equation (DE) is an equation containing some unknown function(s) and their (possibly higher order) derivatives.

- It is p-th order if the highest order derivative involved has order p.

$$\frac{dx}{dt} = x^2 + 1 \qquad \text{(1st order)}$$

$$\frac{d^3x}{dt^3} + \left(\frac{dx}{dt}\right)^2 = x - t \qquad \text{(3rd order)}$$

- A (particular) solution to a DE is a formula of unknown function(s) so that the DE is satisfied.

- The general solution to a DE is a formula of unknown functions (involving parameters) which describe all sol'ns to the DE.

General form of a first order DE w/ one unknown func $y(t)$

$$\frac{dy}{dt} = f(t, y) \quad \text{. - - - - - - - - - -} \quad (*)$$

Initial value problem : $(*)$ w/ initial condition $y(t_0) = y_0$

"Usually" such an initial value problem has a unique sol'n.

Ex Verify that $x(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$ is a sol'n to

$$m \frac{d^2x}{dt^2} = -kx \qquad (\text{where } k, m > 0)$$

$$\frac{dx}{dt} = -\sin\left(\sqrt{\frac{k}{m}}\, t\right) \cdot \sqrt{\frac{k}{m}}$$

$$\frac{d^2x}{dt^2} = -\cos\left(\sqrt{\frac{k}{m}}\, t\right) \cdot \sqrt{\frac{k}{m}} \cdot \sqrt{\frac{k}{m}}$$

$$m \frac{d^2x}{dt^2} = -\cos\left(\sqrt{\frac{k}{m}}\, t\right) \cdot k \qquad -kx = -k \cdot \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

How to study DE ?

- Find explicit sol'ns (only possible for some special cases)
- Numerical sol'ns: get approximations of sol'ns via algorithms.
- Qualitative analysis.