1.2 Separation of variables

Recall: general first order DE in $y(t)$:

$$\frac{dy}{dt} = f(t, y)$$

- A first order DE in $y(t)$ is separable if it has the form

$$\frac{dy}{dt} = g(t)\,h(y)$$

$$dy = g(t)\,h(y)\,dt$$

$$\frac{dy}{h(y)} = g(t)\,dt$$

$$\int \frac{dy}{h(y)} = \int g(t)\,dt$$

Ex Find general sol'n: $\frac{dy}{dt} = \underbrace{\frac{t^2}{y^3}}_{t^2 \cdot \frac{1}{y^3}}$

$$y^3\,dy = t^2\,dt$$

$$\int y^3\,dy = \int t^2\,dt$$

$$\cancel{\frac{1}{4}y^4 + C_1 = \frac{1}{3}t^3 + C_2}$$

$$\frac{1}{4}y^4 = \frac{1}{3}t^3 + C \longrightarrow \text{keep one constant } C$$

$$y^4 = 4\left(\frac{1}{3}t^3 + C\right)$$

$$y = \pm\left(4\left(\frac{1}{3}t^3 + C\right)\right)^{\frac{1}{4}}$$

Ex Solve initial value problem

$$\frac{dy}{dt} = (y^2+1)t, \qquad y(0) = 1$$

$$\int \frac{dy}{y^2+1} = \int t \, dt$$

$$\tan^{-1}(y) = \frac{1}{2}t^2 + C$$

initial condition $\rightsquigarrow$ $\tan^{-1}(1) = \frac{1}{2}\cdot 0^2 + C$

$$C = \frac{\pi}{4}$$

$$\tan^{-1}(y) = \frac{1}{2}t^2 + \frac{\pi}{4}$$

$$y = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)$$

- In a separable DE, $\frac{dy}{dt} = g(t)h(y)$, if $h(y) = 0$ has a sol'n $y = y_*$, then the constant function $y(t) = y_*$ is also a sol'n to the DE (Missing sol'n)

Ex $\frac{dy}{dt} = (y-1)^2 \cos t$ general sol'n

$$\int \frac{dy}{(y-1)^2} = \int \cos t \, dt$$

Missing sol'n : $(y-1)^2 = 0$ $\quad y = 1$

$$-\frac{1}{y-1} = \sin t + C$$

$$\int \frac{dy}{(y-1)^2} \qquad u = y-1, \quad du = dy$$

$$= \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C = -\frac{1}{y-1} + C$$

$$\frac{1}{y-1} = -\sin t - C$$

$$y-1 = \frac{1}{-\sin t - C}$$

$$y = \frac{1}{-\sin t - C} + 1$$

general sol'n : $y = \dfrac{1}{-\sin t - C} + 1$ or $y = 1$

- Sometimes after taking integral, y cannot be solved from the algebraic eq. One can only express the sol'n as an implicit function.

<u>Ex</u> $\dfrac{dy}{dt} = \dfrac{t}{y^4+1}$ general sol'n.

$$\int (y^4+1)\,dy = \int t\,dt$$

$$\underbrace{\frac{1}{5}y^5 + y}_{} = \frac{1}{2}t^2 + C$$

Cannot solve for y. Leave as implicit function

- A DE of the form $\dfrac{dy}{dt} = h(y)$ is called an <u>autonomous DE</u>

It is a special Case of separable DE.

Ex Solve exponential growth model $\frac{dP}{dt} = kP \quad (k>0)$

$$\int \frac{dP}{P} = \int k\,dt$$

Missing sol'n : $P = 0$

$$\ln|P| = kt + C_1$$

$$|P| = e^{kt+C_1} = e^{C_1} e^{kt}$$

$$P = \underbrace{\pm e^{C_1}}_{\text{an arbitrary nonzero number}} e^{kt}$$

$\longrightarrow$ general sol'n: $P = Ce^{kt}$

- Mixing problems

Ex A vat initially contains 10 gallons pure water.
Salty water w/ concentration 3 oz/gallon flows in at a rate of 2 gallon/sec. Well-mixed liquid flows out to make the liquid volume in the vat unchanged

What is the concentration of salt in the vat after 4 sec ?

Denote the amount of salt in the vat as $S(t)$

Concentration of salt in the vat : $\frac{S}{10}$

$$\frac{dS}{dt} = \underbrace{3}_{\text{in-flow}} \times 2 - \underbrace{\frac{S}{10} \times 2}_{\text{out-flow}} = 6 - \frac{1}{5}S, \quad S(0) = 0$$

$$\int \frac{dS}{6-\frac{1}{5}S} = \int dt$$

this denom. $\neq 0$ at initial value
⇝ don't need missing sol'n.

$$-5\ln\left|6-\tfrac{1}{5}S\right| = t + C$$

$$\int \frac{dS}{6-\frac{1}{5}S} \qquad u = 6-\tfrac{1}{5}S \qquad du = -\tfrac{1}{5}\,dS$$

$$= -5\int \frac{du}{u}$$

$$= -5\ln|u| + C$$

$$= -5\ln\left|6-\tfrac{1}{5}S\right| + C$$

initial condition ⇝

$$-5\ln\left|6-\tfrac{1}{5}\cdot 0\right| = 0 + C$$

$$C = -5\ln 6$$

$$-5\ln\left(6-\tfrac{1}{5}S\right) = t - 5\ln 6$$

$$\ln\left|\tfrac{1}{5}S-6\right| \;\; \text{≀} \;\; \ln\left(6-\tfrac{1}{5}S\right)$$

$$\ln\left(6-\tfrac{1}{5}S\right) = -\tfrac{1}{5}t + \ln 6$$

$$6-\tfrac{1}{5}S = e^{-\frac{1}{5}t}e^{\ln 6} = 6e^{-\frac{1}{5}t}$$

$$-\tfrac{1}{5}S = 6e^{-\frac{1}{5}t} - 6$$

$$S = -30e^{-\frac{1}{5}t} + 30$$

$$\Rightarrow \quad \frac{S(4)}{10} = \frac{-30e^{-\frac{4}{5}} + 30}{10} = -3e^{-\frac{4}{5}} + 3.$$

Ex general sol'n $\quad \dfrac{dy}{dt} = ty + 2t$

$$\frac{dy}{dt} = t(y+2)$$

$$\int \frac{dy}{y+2} = \int t\,dt$$

Missing sol'n: $y+2=0$

$$y = -2$$

$$\ln|y+2| = \frac{1}{2}t^2 + C_1$$

$$|y+2| = e^{C_1} e^{\frac{1}{2}t^2}$$

$$y+2 = \pm e^{C_1} e^{\frac{1}{2}t^2}$$

$$y = \underbrace{\pm e^{C_1}}_{C} e^{\frac{1}{2}t^2} - 2$$

$\Rightarrow$ general sol'n $\quad y = C e^{\frac{1}{2}t^2} - 2$