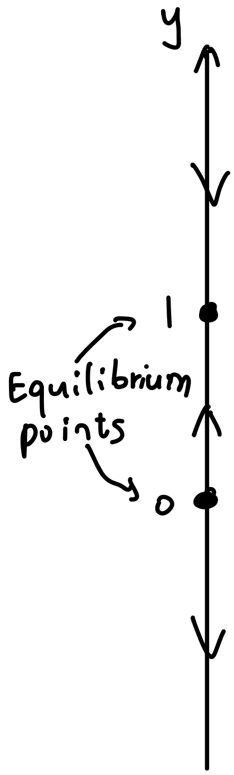
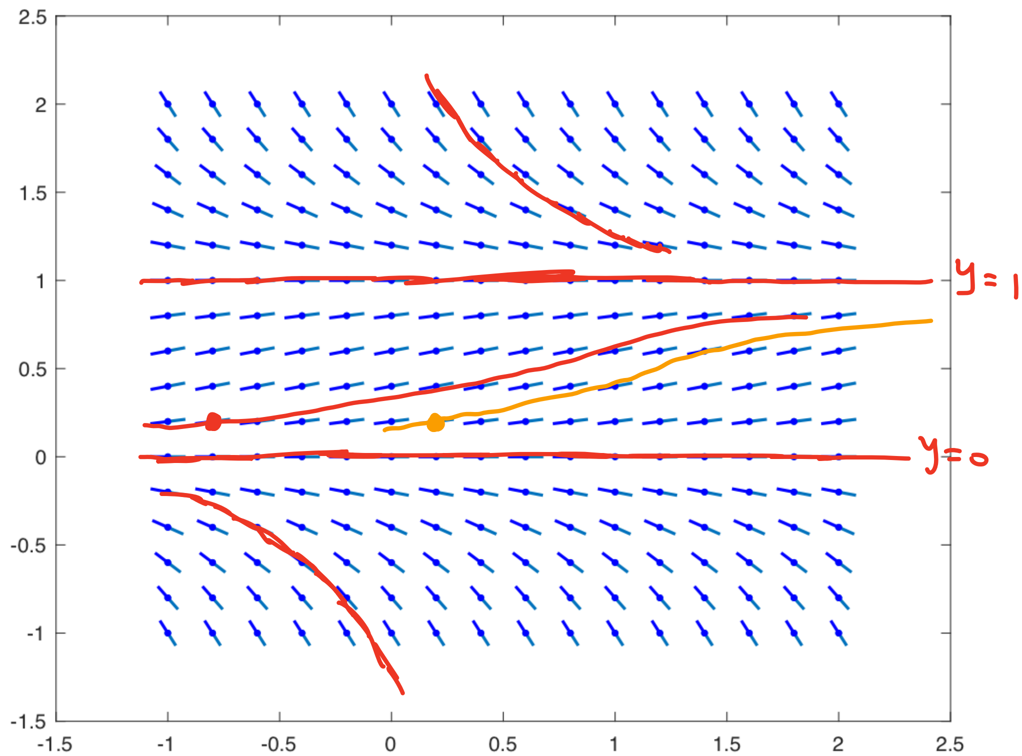
1.6 (continued)

Ex $\frac{dy}{dt} = y(1-y)$





- The phase line for autonomous DE $\frac{dy}{dt} = f(y)$:

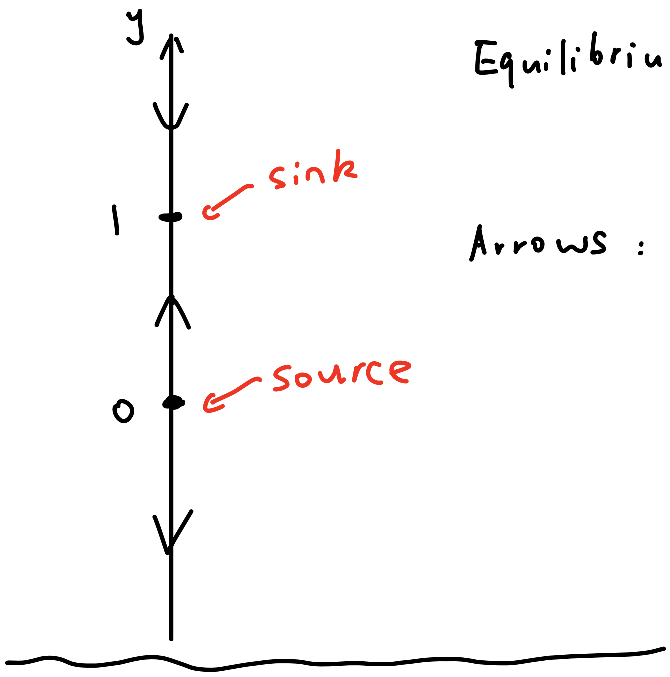
A vertical y-axis containing the following:

① All equilibrium points (that are, where $f(y) = 0$)

② These pts separate the y-axis into intervals. For each interval:

$$\begin{cases} \text{If } f(y) > 0, \text{ draw } \text{``}\uparrow\text{''} \\ \text{If } f(y) < 0, \text{ draw } \text{``}\downarrow\text{''} \end{cases}$$

Ex $\frac{dy}{dt} = y(1-y)$ Sketch phase line



Equilibrium pts: $y(1-y)=0$

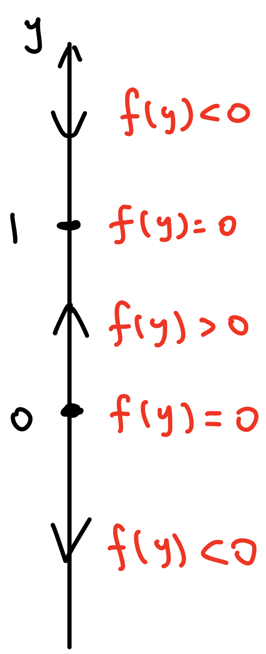
$\Rightarrow y = 0, 1$

Arrows: $f(-1) = (-1)\cdot(2) = -2 < 0$

$f(\frac{1}{2}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} > 0$

$f(2) = 2\cdot(-1) = -2 < 0$

Information we obtain from the phase line:



- If initial value is 0 or 1, then sol'n is constant
- If initial value is in $(-\infty, 0)$, then
  $\begin{cases} \text{sol'n is decreasing} \\ \text{approaching } -\infty \text{ as } t \text{ increases} \end{cases}$
- If initial value is in $(0,1)$, then
  $\begin{cases} \text{sol'n is increasing} \\ \text{approaching } 1 \text{ as } t \to \infty \end{cases}$
- If initial value is in $(1, \infty)$, then
  $\begin{cases} \text{sol'n is decreasing} \\ \text{approaching } 1 \text{ as } t \to \infty \end{cases}$

Ex Sketch phase line :

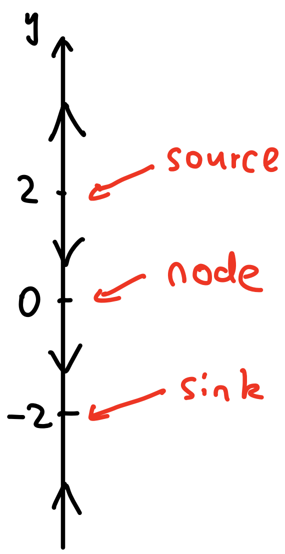
$$\frac{dy}{dt} = \frac{e^{-y}(y^4 - 4y^2)}{y^2+2}$$

Determine behavior of sol'ns to initial value problems w/

① $y(0) = 1$

② $y(0) = -3$

(inc./dec. , approaches ? as t increases)



Equilibrium pts :

$$\frac{e^{-y}(y^4 - 4y^2)}{y^2+2} = 0$$

$$y^4 - 4y^2 = 0 \qquad y^2(y^2 - 4) = 0$$

$$y^2(y+2)(y-2) = 0$$

$$y = 0, \ -2, \ 2$$

Arrows : $f(y) = \frac{\overset{+}{e^{-y}}\,\overset{+}{y^2}\,(y^2-4)}{\underset{+}{\boxed{y^2+2}}}$

At $y = -3$: $(-3)^2 - 4 = 5 > 0$

At $y = -1$: $(-1)^2 - 4 = -3 < 0$

At $y = 1$: $1^2 - 4 = -3 < 0$

At $y = 3$: $3^2 - 4 = 5 > 0$

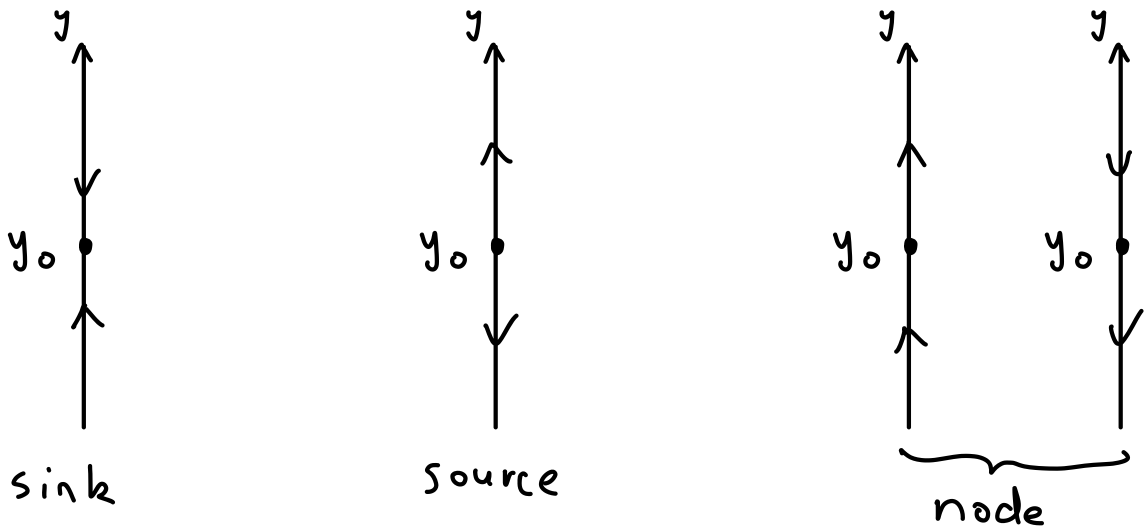
Initial value problem w/ $y(0) = 1$: dec. , approaches 0

Initial value problem w/ $y(0) = -3$: inc. , approaches -2

Classification of equilibrium pts

Let y_0 be an equilibrium pt.

- We say y_0 is a <u>sink</u> if any sol'n w/ initial value sufficiently close to y_0 will approach y_0 as t increases.
- We say y_0 is a <u>source</u> if any sol'n w/ initial value sufficiently close to y_0 will get away from y_0 as t increases.
- Otherwise, we say y_0 is a <u>node</u>



<u>Thm</u> Let y_0 be an equilibrium pt of $\frac{dy}{dt} = f(y)$

- If $f'(y_0) < 0$, then y_0 is a sink
- If $f'(y_0) > 0$, then y_0 is a source

- If $f'(y_0) = 0$, then no info is obtained.

Ex $\dfrac{dy}{dt} = \underbrace{\sin y + y^2 - \pi^2}_{f(y)}$

① Verify that $y = \pi$ is an equilibrium pt

② Determine its type

$f(\pi) = \sin\pi + \pi^2 - \pi^2 = 0 \Rightarrow y = \pi$ is an equilibrium pt

$f'(y) = \cos y + 2y$

$f'(\pi) = \cos\pi + 2\pi = -1 + 2\pi > 0$

$\Rightarrow y = \pi$ is a source.

1.7 Bifurcations

Consider an autonomous DE w/ a parameter μ

$\dfrac{dy}{dt} = f_\mu(y)$ (for example, $\frac{dy}{dt} = y^2 - \mu$)

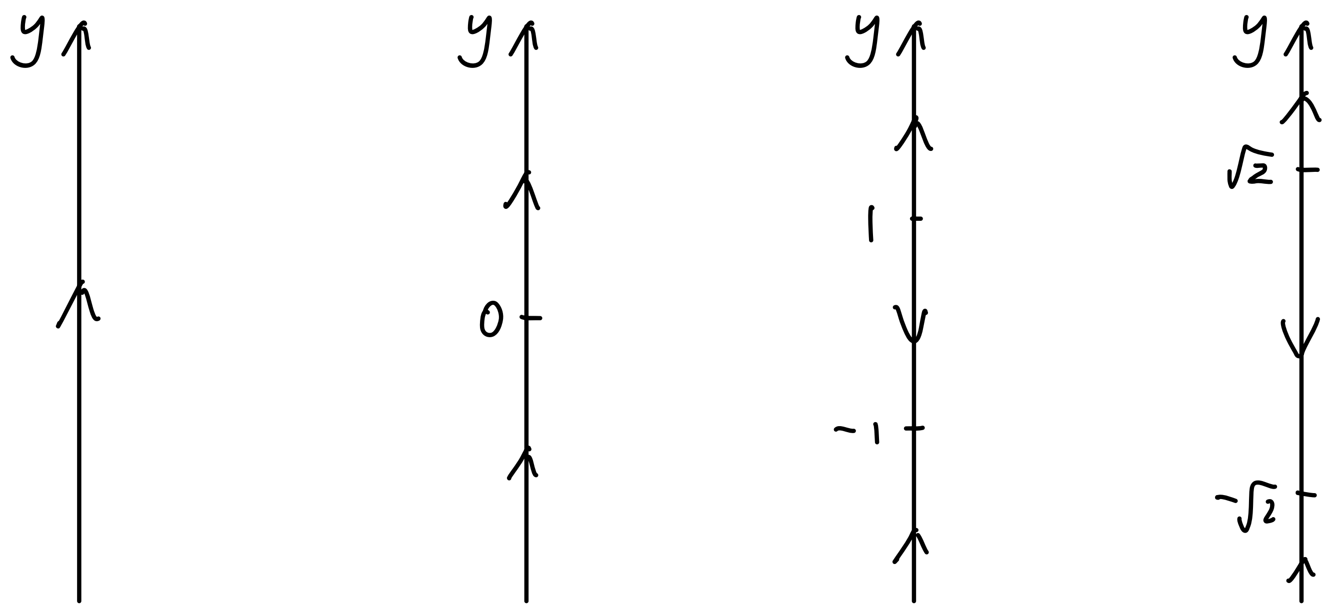
For different values of μ, check the qualitative behavior of its phase line.

$\mu = -1$, $\frac{dy}{dt} = y^2 + 1$

$\mu = 0$ bifurcation value, $\frac{dy}{dt} = y^2$

$\mu = 1$, $\frac{dy}{dt} = y^2 - 1$

$\mu = 2$, $\frac{dy}{dt} = y^2 - 2$



- A value of μ around which the qualitative behavior of the phase line changes is called a <u>bifurcation value</u>.