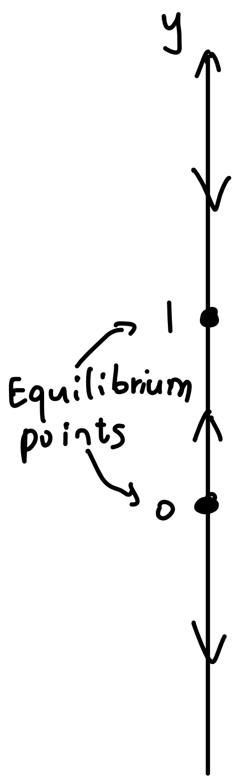
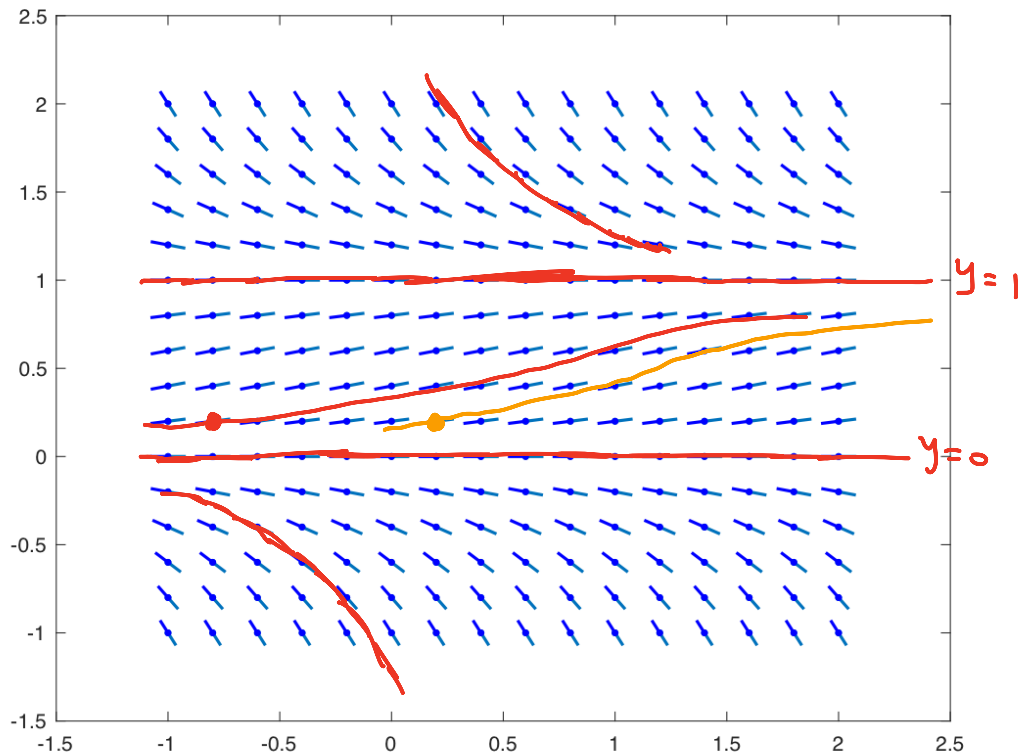
1.6 (continued)

Ex $\dfrac{dy}{dt} = y(1-y)$





- The <u>phase line</u> for autonomous DE $\dfrac{dy}{dt} = f(y)$:

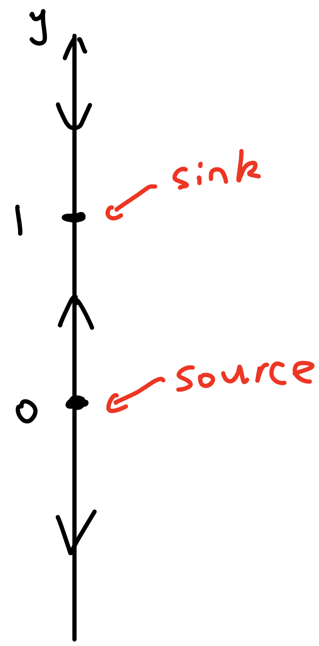
  A vertical y-axis containing the following:

  ① All <u>equilibrium points</u> (that are, where $f(y)=0$)

  ② These pts separate the y-axis into intervals. For each interval:

  $\begin{cases} \text{If } f(y) > 0, \text{ draw } \text{“}\uparrow\text{”} \\ \text{If } f(y) < 0, \text{ draw } \text{“}\downarrow\text{”} \end{cases}$

Ex $\dfrac{dy}{dt} = y(1-y)$ Sketch phase line



Equilibrium pts : $y(1-y) = 0$

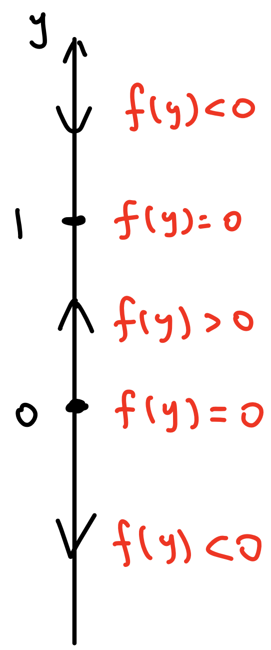
$\Rightarrow y = 0, 1$

Arrows : $f(-1) = (-1)\cdot(2) = -2 < 0$

$f(\tfrac{1}{2}) = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} > 0$

$f(2) = 2\cdot(-1) = -2 < 0$

Information we obtain from the phase line :



- If initial value is 0 or 1, then sol'n is constant
- If initial value is in $(-\infty, 0)$, then $\begin{cases} \text{sol'n is decreasing} \\ \text{approaching } -\infty \text{ as } t \text{ increases} \end{cases}$
- If initial value is in $(0, 1)$, then $\begin{cases} \text{sol'n is increasing} \\ \text{approaching } 1 \text{ as } t \to \infty \end{cases}$
- If initial value is in $(1, \infty)$, then $\begin{cases} \text{sol'n is decreasing} \\ \text{approaching } 1 \text{ as } t \to \infty \end{cases}$

Ex Sketch phase line:

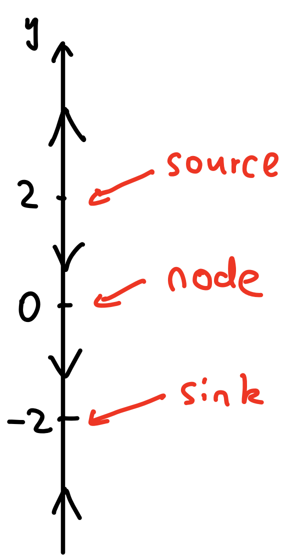
$$\frac{dy}{dt} = \frac{e^{-y}(y^4 - 4y^2)}{y^2 + 2}$$

Determine behavior of sol'ns to initial value problems w/

① $y(0) = 1$

② $y(0) = -3$

(inc./dec., approaches ? as t increases)



Equilibrium pts:

$$\frac{e^{-y}(y^4 - 4y^2)}{y^2 + 2} = 0$$

$$y^4 - 4y^2 = 0 \qquad y^2(y^2 - 4) = 0$$

$$y^2(y+2)(y-2) = 0$$

$$y = 0, -2, 2$$

Arrows: $f(y) = \dfrac{\overset{+}{e^{-y}}\,\overset{+}{y^2}(y^2-4)}{\underset{+}{y^2+2}}$

At $y = -3$: $(-3)^2 - 4 = 5 > 0$

At $y = -1$: $(-1)^2 - 4 = -3 < 0$

At $y = 1$: $1^2 - 4 = -3 < 0$

At $y = 3$: $3^2 - 4 = 5 > 0$

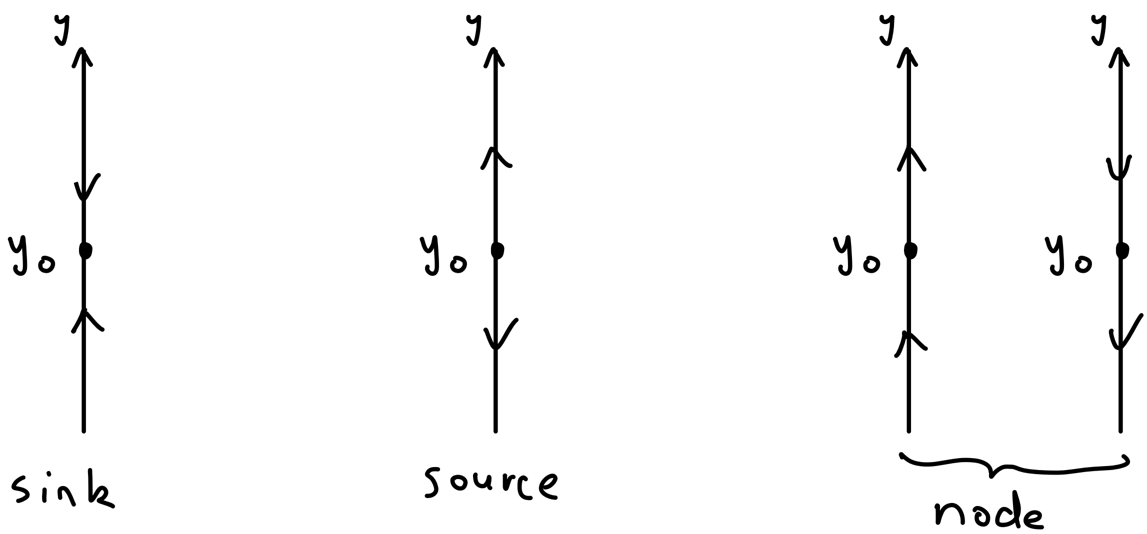
Initial value problem w/ $y(0) = 1$: dec., approaches 0

Initial value problem w/ $y(0) = -3$: inc., approaches -2

Classification of equilibrium pts

Let y_0 be an equilibrium pt.

- We say y_0 is a <u>sink</u> if any sol'n w/ initial value sufficiently close to y_0 will approach y_0 as t increases.
- We say y_0 is a <u>source</u> if any sol'n w/ initial value sufficiently close to y_0 will get away from y_0 as t increases.
- Otherwise, we say y_0 is a <u>node</u>



<u>Thm</u> Let y_0 be an equilibrium pt of $\frac{dy}{dt} = f(y)$

- If $f'(y_0) < 0$, then y_0 is a sink
- If $f'(y_0) > 0$, then y_0 is a source

- If $f'(y_0) = 0$, then no info is obtained.

Ex $\dfrac{dy}{dt} = \underbrace{\sin y + y^2 - \pi^2}_{f(y)}$

① Verify that $y = \pi$ is an equilibrium pt

② Determine its type

$f(\pi) = \sin\pi + \pi^2 - \pi^2 = 0 \Rightarrow y = \pi$ is an equilibrium pt

$f'(y) = \cos y + 2y$

$f'(\pi) = \cos\pi + 2\pi = -1 + 2\pi > 0$

$\Rightarrow y = \pi$ is a source.

1.7 Bifurcations

Consider an autonomous DE w/ a parameter μ

$\dfrac{dy}{dt} = f_\mu(y)$ (for example, $\dfrac{dy}{dt} = y^2 - \mu$)

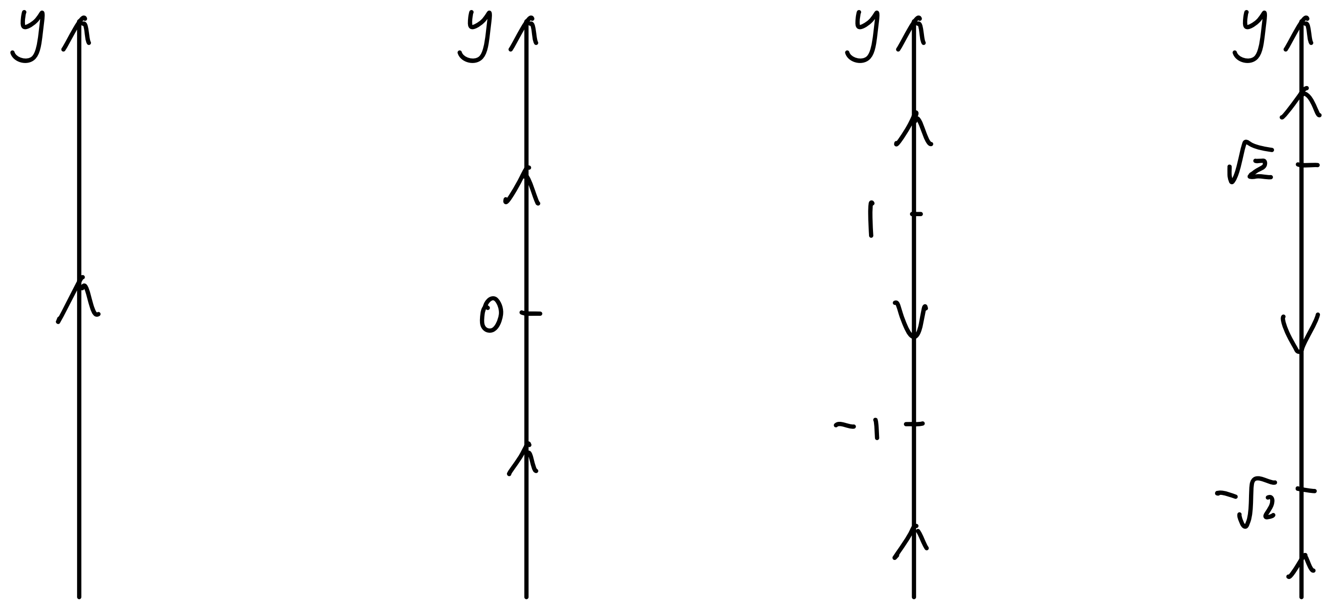
For different values of μ, check the qualitative behavior of its phase line.

$\mu = -1$ $\frac{dy}{dt} = y^2 + 1$

$\mu = 0$ bifurcation value $\frac{dy}{dt} = y^2$

$\mu = 1$ $\frac{dy}{dt} = y^2 - 1$

$\mu = 2$ $\frac{dy}{dt} = y^2 - 2$



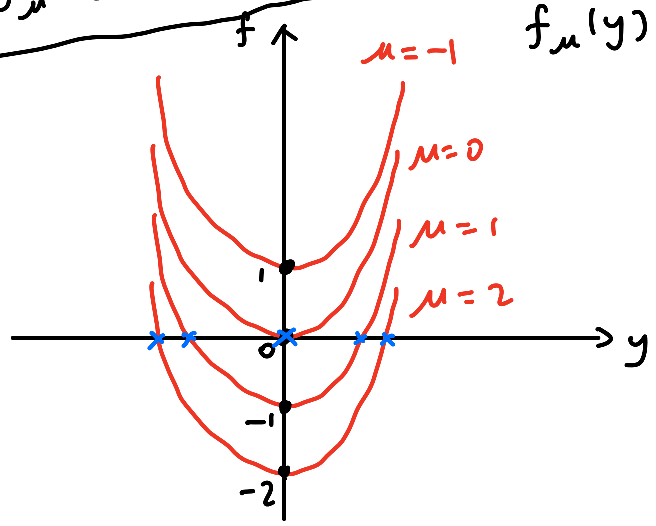
- A value of μ around which the qualitative behavior of the phase line changes is called a <u>bifurcation value</u>

1.7 (continued)

Fact: for a bifurcation value μ, the corresponding $f_\mu(y)$ must have a multiple root, that is, some y_* satisfying

$$f_\mu(y_*) = f'_\mu(y_*) = 0$$

$f_\mu(y) = y^2 - \mu$



- Suppose $f_\mu(y)$ is a quadratic function in y. then $f_\mu(y)$ has a multiple root if and only if the discriminant "$b^2 - 4ac$" $= 0$

Applied to $f_\mu(y) = y^2 - \mu = \underbrace{1}_{a} \cdot y^2 + \underbrace{0}_{b} \cdot y + \underbrace{(-\mu)}_{c}$

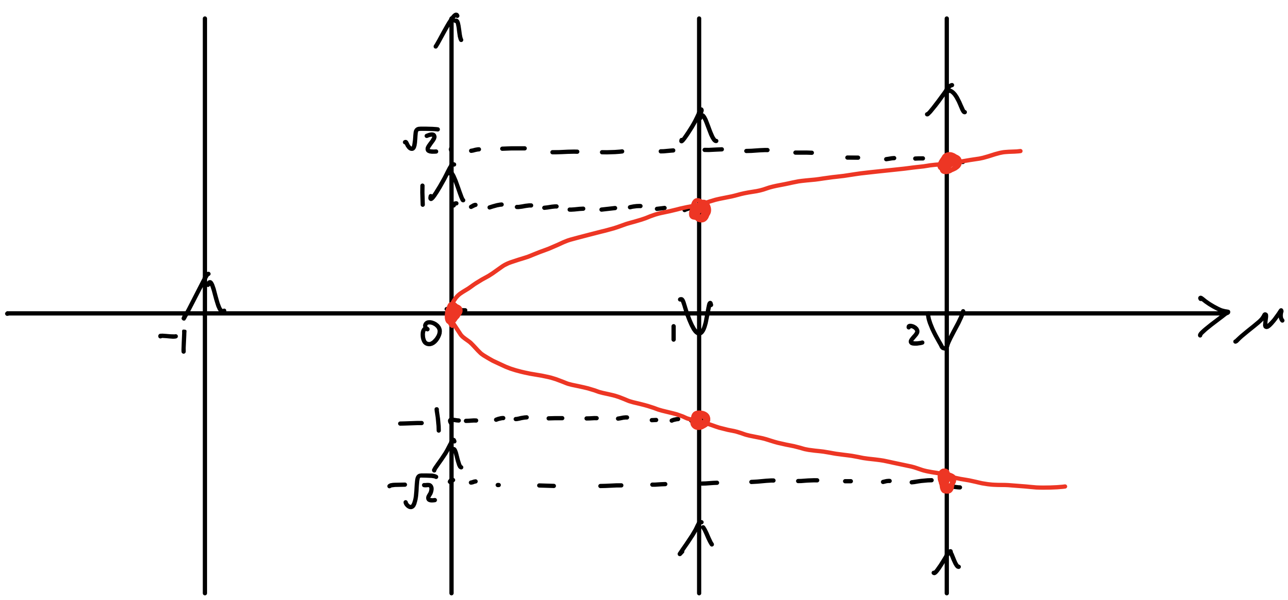
$$b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-\mu) = 0$$

$$4\mu = 0 \qquad \mu = 0$$

- Bifurcation diagram : sketch phase lines in (μ, y)-plane

Connect all equilibrium pts for every μ as a curve

Ex $\frac{dy}{dt} = y^2 - \mu$



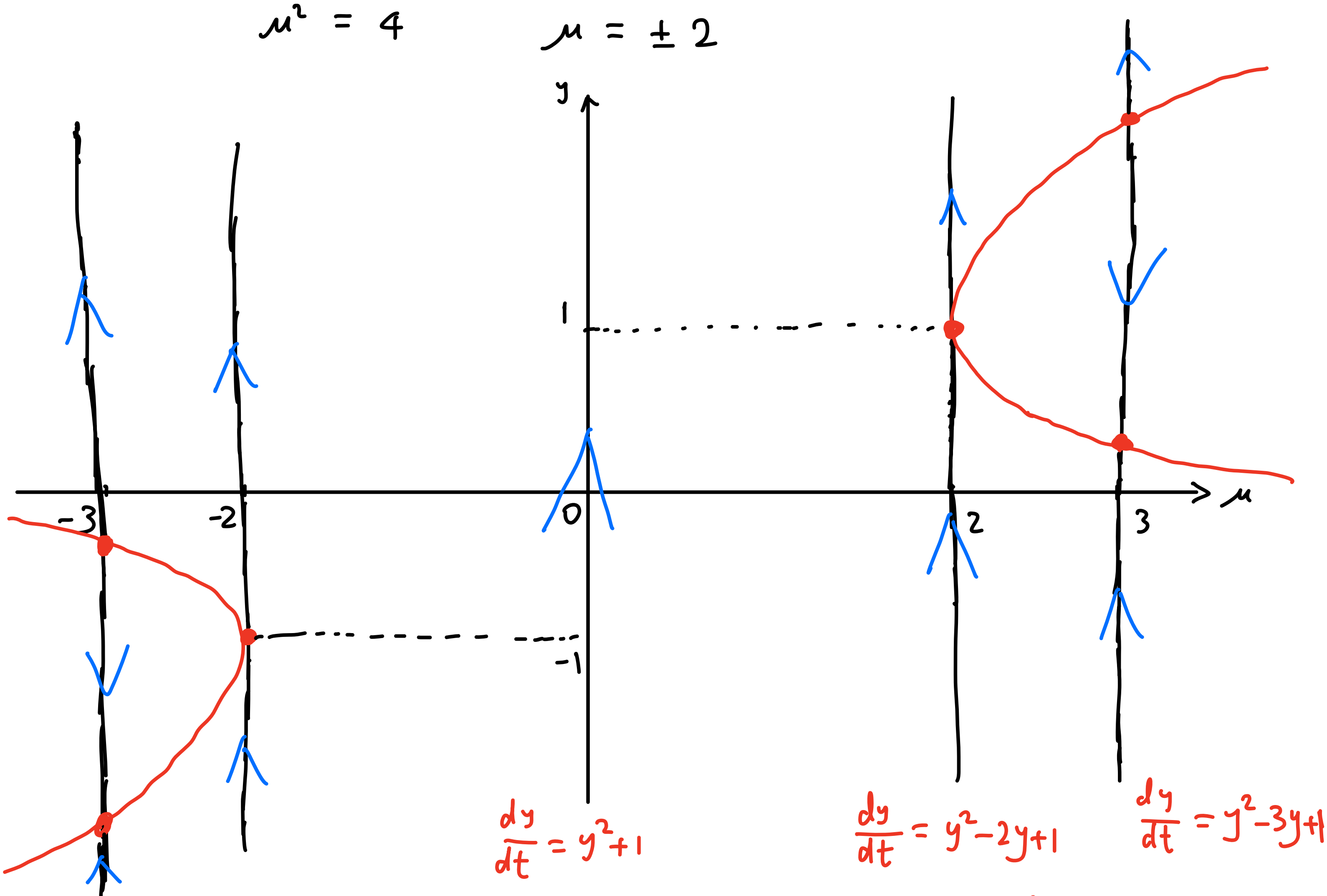
<u>Ex</u> $\dfrac{dy}{dt} = \underbrace{y^2}_{a} \underbrace{-\mu y}_{b} \underbrace{+1}_{c}$ Find bifurcation values.

sketch bifurcation diagram

$$b^2 - 4ac = (-\mu)^2 - 4 \cdot 1 \cdot 1 = 0$$

$$\mu^2 - 4 = 0$$

$$\mu^2 = 4 \qquad \mu = \pm 2$$



$\dfrac{dy}{dt} = y^2 + 1$

$\dfrac{dy}{dt} = y^2 - 2y + 1$
$= (y-1)^2$

$\dfrac{dy}{dt} = y^2 - 3y + 1$

$$y^2 - 3y + 1 = 0$$

$$y = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

1.8, 1.9 Linear equations

Def A first order DE for $y(t)$ is <u>linear</u> if it has the form

$$\frac{dy}{dt} = a(t)\,y + b(t)$$

- Solve linear DE by <u>integrating factors</u>

Denote $g(t) = -a(t)$

$$\frac{dy}{dt} + g(t)\,y = b(t)$$

write the linear DE in this form

Multiply by the <u>integrating factor</u>

$$\boxed{\mu(t) = e^{\int g(t)\,dt}}$$

$$\frac{d\mu}{dt} = e^{\int g(t)\,dt} \cdot g(t)$$

$$= \mu(t)\,g(t)$$

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)\,y = \mu(t)b(t)$$

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}\,y = \mu(t)b(t)$$

$$\frac{d}{dt}\left(\mu(t)\cdot y\right) = \mu(t)\,b(t)$$

$$\mu(t)\cdot y = \int \mu(t)\,b(t)\,dt$$

$$y = \frac{1}{\mu(t)}\int \mu(t)\,b(t)\,dt$$

Ex Solve $\frac{dy}{dt} + \underbrace{\frac{1}{t}}_{} y = t^2$

$\hookrightarrow g(t) = \frac{1}{t}$

- When calculating $\mu(t)$, if it involves $\ln|\cdots|$, remove the absolute value.
- When calculating $\mu(t)$, you don't need "$+C$"

$$\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$

$$\frac{d}{dt}(t\,y) = t^2 \cdot t = t^3$$

$$t\,y = \int t^3 dt = \frac{1}{4}t^4 + C$$

$$y = \frac{1}{4}t^3 + \frac{C}{t}$$

Ex Solve $\frac{dy}{dt} = \frac{1}{t}y + t$, $y(1) = 2$

$$\frac{dy}{dt} \underbrace{- \frac{1}{t}}_{} y = t$$

$\hookrightarrow g(t) = -\frac{1}{t}$

$$\mu(t) = e^{\int -\frac{1}{t} dt} = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1}$$

$$\frac{d}{dt}(t^{-1} y) = t \cdot t^{-1} = 1$$

$$t^{-1} y = \int 1\, dt = t + C$$

initial condition $\rightsquigarrow$ $1^{-1} \cdot 2 = 1 + C$ $\qquad C = 1$

$$t^{-1} y = t + 1$$

$$y = t^2 + t$$

$\underline{\text{Ex}}$ Solve $\frac{dy}{dt} = ty + t^3$

$$\frac{dy}{dt} \underbrace{- t}_{} y = t^3$$

$g(t) = -t$

$$\mu(t) = e^{\int -t \, dt} = e^{-\frac{1}{2}t^2}$$

$$\frac{d}{dt}\left(e^{-\frac{1}{2}t^2} y\right) = t^3 e^{-\frac{1}{2}t^2}$$

$$e^{-\frac{1}{2}t^2} y = \int t^3 e^{-\frac{1}{2}t^2} dt$$