Total time: 75 minutes.
Total points: 100.

Problem 1 ($10 \times 2 = 20$ points). Find the general solution to the following differential equations:

$$(1) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^4 + \frac{2}{t}y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{2}{t}y = 3t^4$$

$$g(t) = -\frac{2}{t}, \quad \mu(t) = e^{\int -\frac{2}{t}\,\mathrm{d}t} = e^{-2\ln t} = t^{-2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^{-2}y) = 3t^2$$

$$t^{-2}y = \int 3t^2\,\mathrm{d}t = t^3 + C$$

$$y = t^5 + Ct^2$$

$$(2) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y^2+2y+1}{t}$$

$$\int \frac{\mathrm{d}y}{(y+1)^2} = \int \frac{1}{t}\,\mathrm{d}t$$

(missing solution: $y = -1$)

$$-\frac{1}{y+1} = \ln|t| + C$$

$$\frac{1}{y+1} = -(\ln|t| + C)$$

$$y + 1 = -\frac{1}{\ln|t| + C}$$

$$y = -\frac{1}{\ln|t| + C} - 1$$

The answer is

$$y = -\frac{1}{\ln|t| + C} - 1 \text{ or } y = -1$$

Problem 2 ($10+10=20$ points). Consider the initial value problems:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 t, \quad y(1) = 2$$

(1) Find its solution.

$$\int \frac{\mathrm{d}y}{y^2} = \int t\,\mathrm{d}t$$

(we don't need the missing solution because initial value of y is 2, not making the denominator zero.)

$$-\frac{1}{y} = \frac{1}{2}t^2 + C$$

Using initial condition,

$$-\frac{1}{2} = \frac{1}{2}1^2 + C, \quad C = -1$$

$$-\frac{1}{y} = \frac{1}{2}t^2 - 1$$

$$\frac{1}{y} = -\frac{1}{2}t^2 + 1$$

$$y = \frac{1}{-\frac{1}{2}t^2 + 1}$$

(2) Use Euler's method with $\Delta t = 0.1$ to approximate $y(1.2)$.

$$t_0 = 1, \quad t_1 = 1.1, \quad t_2 = 1.2$$

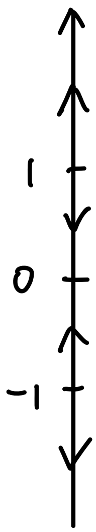
(need 2 iterations)

$$y_1 = y_0 + \Delta t f(t_0, y_0) = 2 + 0.1 \times 2^2 \times 1 = 2.4$$

$$y_2 = y_1 + \Delta t f(t_1, y_1) = 2.4 + 0.1 \times 2.4^2 \times 1.1 = 3.0336$$

Problem 3 ($10 + 5 + 5 = 20$ points). Consider the autonomous differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{2y^2}(y^5 - y)$$



(1) Sketch the phase line.
Equilibrium points:

$$e^{2y^2}(y^5 - y) = 0, \quad y^5 - y = 0, \quad y(y^4 - 1) = 0, \quad y = 0, 1, -1$$

Check special values (only need to do it for $y^5 - y$):

$$2^5 - 2 > 0, \quad 0.5^5 - 0.5 < 0, \quad (-0.5)^5 - (-0.5) > 0, \quad (-2)^5 - (-2) < 0$$

(2) Determine the types of all equilibrium point(s) in (1).
1: source; 0: sink; -1: source

(3) Describe the qualitative property of the solution with initial condition $y(0) = -0.5$. (increasing/decreasing, where does it approach as t increases)
increasing, approaching 0.

Problem 4 ($10 + 10 = 20$ points). Consider the damped harmonic oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y}{\mathrm{d}t} + 4y = 0$$

(1) Find its general solution.

$$s^2 + 5s + 4 = 0$$

$$(s+1)(s+4) = 0$$

$$s_1 = -1, \quad s_2 = -4$$

general solution:

$$y(t) = C_1 e^{-t} + C_2 e^{-4t}$$

(2) Use your result in (1) to solve the initial value problem with initial condition

$$y(0) = 3, \; y'(0) = -7$$

$$y'(t) = -C_1 e^{-t} - 4C_2 e^{-4t}$$

Using initial condition,

$$C_1 + C_2 = 3, \quad -C_1 - 4C_2 = -7$$

Solve to get

$$C_1 = \frac{5}{3}, \quad C_2 = \frac{4}{3}$$

The answer is

$$y(t) = \frac{5}{3}e^{-t} + \frac{4}{3}e^{-4t}$$

Problem 5 ($12+4+4=20$ points). A vat initially contains 10 gallons pure water. Salty water with concentration 3 oz/gal flows in at a rate of 2 gal/min. Well-mixed liquid flows out at a rate of 3 gal/min.

(1) Find the amount of salt in the vat, as a function of time t.
Net out-flow rate: $3-2=1$. Volume of liquid in the vat: $V(t)=10-t$.
Let $S(t)$ be the amount of salt in the vat. Then the concentration of salt in the vat is $\frac{S}{10-t}$.

$$\frac{dS}{dt}=3\times 2-\frac{S}{10-t}\times 3=-\frac{3}{10-t}S+6, \quad S(0)=0$$

$$\frac{dS}{dt}+\frac{3}{10-t}S=6$$

$$g(t)=\frac{3}{10-t}, \quad \mu(t)=e^{\int \frac{3}{10-t}\,dt}=e^{-3\ln(10-t)}=(10-t)^{-3}$$

$$\frac{d}{dt}((10-t)^{-3}S)=6(10-t)^{-3}$$

$$(10-t)^{-3}S=\int 6(10-t)^{-3}\,dt=3(10-t)^{-2}+C$$

Using initial condition,

$$0=3\times 10^{-2}+C, \quad C=-0.03$$

$$(10-t)^{-3}S=3(10-t)^{-2}-0.03$$

$$S=3(10-t)-0.03(10-t)^3$$

(2) When the vat contains 4 gallons liquid, what is the concentration of salt in the vat?
When $V(t)=10-t=4$, $t=6$. Concentration is

$$\frac{S(6)}{4}=\frac{3\times 4-0.03\times 4^3}{4}=2.52$$

(3) When the vat contains 4 gallons liquid, is the amount of salt in the vat increasing or decreasing? Justify you answer.
Again, at $t=6$. The DE gives

$$\frac{dS}{dt}(6)=-\frac{3}{4}S(6)+6=-3\times 2.52+6=-1.56<0$$

Therefore the amount of salt is decreasing.