## Review

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$(a-\lambda)(d-\lambda) - bc = 0, \qquad \lambda, \quad \vec{v} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} \text{ or } \begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$$

## General sol'n

- real distinct $\lambda_1, \lambda_2$ $\rightsquigarrow$ $\vec{v}_1$, $\vec{v}_2$ $\qquad \vec{Y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2$

- complex $\lambda = \alpha \pm \beta i$

  Take $\lambda = \alpha + \beta i$, $\vec{v}$

  $\Rightarrow$ complex sol'n $\vec{Y}(t) = e^{\lambda t}\vec{v}$

  Take real / imag parts $\rightsquigarrow \vec{Y}_1, \vec{Y}_2$

  $\vec{Y} = C_1\vec{Y}_1 + C_2\vec{Y}_2$

- repeated $\lambda$ $\qquad \vec{v}_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad \vec{v}_1 = (A - \lambda I)\vec{v}_0$

$$\vec{Y}(t) = e^{\lambda t}\vec{v}_0 + t e^{\lambda t}\vec{v}_1$$

Ex
$$\begin{cases} \dfrac{dy_1}{dt} = y_1 + 2y_2 \\ \dfrac{dy_2}{dt} = -4y_1 - 3y_2 \end{cases}$$
general sol'n

$$A = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix}$$

$$(1-\lambda)(-3-\lambda) - 2 \times (-4) = 0$$

$$-3 + 2\lambda + \lambda^2 + 8 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4\times1\times5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

For $\lambda = -1 + 2i$, eigenvector $\vec{v} = \begin{pmatrix} -2 \\ 1-(-1+2i) \end{pmatrix} = \begin{pmatrix} -2 \\ 2-2i \end{pmatrix}$

$\Rightarrow$ Complex sol'n $\vec{Y}(t) = e^{(-1+2i)t} \begin{pmatrix} -2 \\ 2-2i \end{pmatrix}$

$$= e^{-t} e^{2it} \begin{pmatrix} -2 \\ 2-2i \end{pmatrix}$$

$$= e^{-t} \left(\cos(2t) + i \sin(2t)\right) \begin{pmatrix} -2 \\ 2-2i \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} -2\cos(2t) - 2i\sin(2t) \\ 2\cos(2t) + 2i\sin(2t) - 2i\cos(2t) + 2\sin(2t) \end{pmatrix}$$

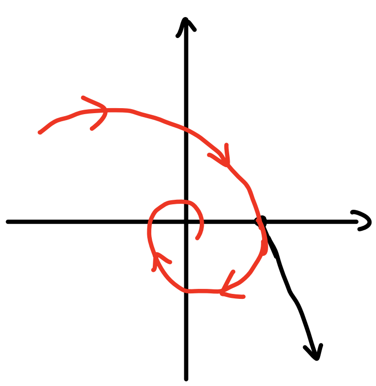
$$= e^{-t} \begin{pmatrix} -2\cos(2t) \\ 2\cos(2t) + 2\sin(2t) \end{pmatrix} + i e^{-t} \begin{pmatrix} -2\sin(2t) \\ 2\sin(2t) - 2\cos(2t) \end{pmatrix}$$

$\Rightarrow$ general sol'n :

$$\vec{Y}(t) = C_1 e^{-t} \begin{pmatrix} -2\cos(2t) \\ 2\cos(2t) + 2\sin(2t) \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -2\sin(2t) \\ 2\sin(2t) - 2\cos(2t) \end{pmatrix}$$

Phase portrait : $\lambda = -1 \pm 2i$ (spiral sink)

$\alpha = -1$

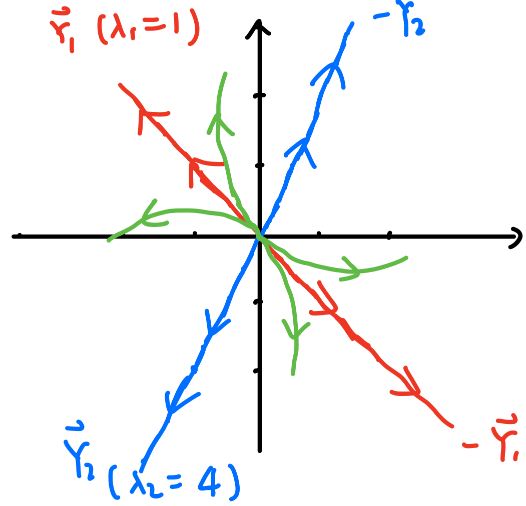


$$\begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Ex Solve initial value problem

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \vec{Y} \qquad \vec{Y}(0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Phase portrait : (source)




$$(2-\lambda)(3-\lambda) - 1 \times 2 = 0$$

$$6 - 5\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 4$$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 2-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ general sol'n $\vec{Y}(t) = C_1 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

initial condition $\rightarrow$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\begin{cases} -2 = -C_1 - C_2 \\ 1 = C_1 - 2C_2 \end{cases}$$

$$-1 = -3C_2 \qquad C_2 = \frac{1}{3}$$

$$C_1 = -C_2 + 2 = -\frac{1}{3} + 2 = \frac{5}{3}$$

$$\vec{Y}(t) = \frac{5}{3} e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{3} e^{4t} \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

Ex $\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \vec{Y}$ general sol'n

$(2-\lambda)(4-\lambda) - 1 \times (-1) = 0$

$8 - 6\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$\lambda = 3$ (repeated)

$A - \lambda I$

$= \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\vec{v}_0 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ $\quad \vec{v}_1 = (A - \lambda I)\vec{v}_0 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix}$

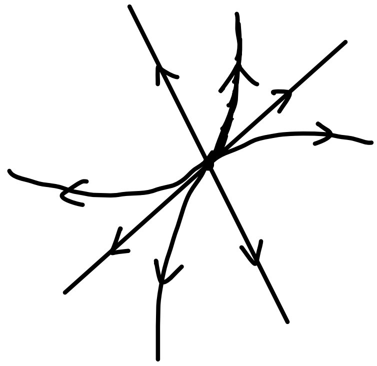
$\Rightarrow$ general sol'n $\vec{Y}(t) = e^{3t}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + t e^{3t}\begin{pmatrix} -C_1 + C_2 \\ -C_1 + C_2 \end{pmatrix}$

# Phase portrait

- real distinct $\lambda_1, \lambda_2$

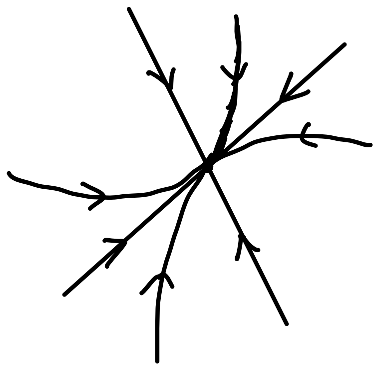
  - $\lambda_1 > 0,\ \lambda_2 < 0$ (saddle)


  - $\lambda_1, \lambda_2 > 0$ (source)



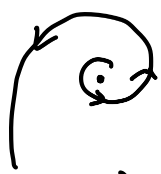
Curves tangent to the line corres. to eigenvalues w/ smaller abs value.

  - $\lambda_1, \lambda_2 < 0$ (sink)



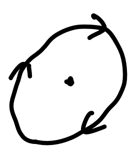
- Complex $\lambda = \alpha \pm \beta i$

  - $\alpha > 0$ (spiral source)



  - $\alpha < 0$ (spiral sink)

  - $\alpha = 0$ (center)



direction:

test vector field $(1, 0)$

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{\nu t}$$

$s^2 + ps + q = 0 \rightsquigarrow$ eigenvalues $\rightsquigarrow y_h$

To find a particular sol'n :

$$\begin{cases} \nu \text{ is not an eigenvalue, try } y(t) = Ce^{\nu t} \\ \nu \text{ is a single eigenvalue, try } y(t) = Cte^{\nu t} \\ \nu \text{ is a repeated eigenvalue, try } y(t) = Ct^2e^{\nu t} \end{cases}$$

<u>Ex</u> $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5y = -2e^{-t}$ general sol'n

$s^2 + 6s + 5 = 0$

$(s+1)(s+5) = 0$

$s_1 = -1, \quad s_2 = -5$

$\Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{-5t}$

To find a particular sol'n , try $y(t) = Cte^{-t}$

$$\frac{dy}{dt} = C(e^{-t} - te^{-t})$$

$$\frac{d^2y}{dt^2} = C(-e^{-t} - e^{-t} + te^{-t}) = C(-2e^{-t} + te^{-t})$$

$$C(-2e^{-t} + te^{-t}) + 6C(e^{-t} - te^{-t}) + 5Cte^{-t} = -2e^{-t}$$

$$C(-2+t) + 6C(1-t) + 5Ct = -2$$

$$C(-2 + t + 6 - 6t + 5t) = -2$$

$$4C = -2$$

$$C = -\frac{1}{2}$$

$\Rightarrow \quad y_p(t) = -\frac{1}{2} t e^{-t}$

$\Rightarrow$ general sol'n $\quad y(t) = -\frac{1}{2} t e^{-t} + C_1 e^{-t} + C_2 e^{-5t}$

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$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t) \quad \text{----} \quad (1)$$

(or $\sin(\omega t)$ )

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Consider

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{i\omega t} \quad \text{-----} \quad (2)$$

Get a particular sol'n to (2), take real part (imaginary) to get a particular sol'n to (1)

Ex Find general sol'n

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \sin(2t) \quad \text{-----} \quad (1)$$

Consider

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{2it} \quad \text{-----} \quad (2)$$

$$s^2 + 2s + 1 = 0$$

$$(s+1)^2 = 0$$

$$s = -1 \quad \text{(repeated)}$$

$$y_h(t) = C_1 e^{-t} + C_2 t e^{-t}$$

To find a particular sol'n to (2), try $y(t) = Ce^{2it}$

$$\frac{dy}{dt} = 2iCe^{2it}$$

$$\frac{d^2y}{dt^2} = -4Ce^{2it}$$

$$-4Ce^{2it} + 2 \times 2iCe^{2it} + Ce^{2it} = e^{2it}$$

$$-4C + 4iC + C = 1$$

$$C(-4 + 4i + 1) = 1$$

$$C = \frac{1}{-3+4i} = \frac{-3-4i}{(-3+4i)(-3-4i)} = \frac{-3-4i}{25}$$

$\Rightarrow$ particular sol'n to (2):

$$y(t) = \frac{1}{25}(-3-4i)e^{2it}$$

$$= \frac{1}{25}(-3-4i)(\cos(2t) + i\sin(2t))$$

$$= \frac{1}{25}(-3\cos(2t) - 3i\sin(2t) - 4i\cos(2t) + 4\sin(2t))$$

Take imaginary part, get particular sol'n to (1):

$$y_p(t) = \frac{1}{25}(-3\sin(2t) - 4\cos(2t))$$

$\Rightarrow$ general sol'n to (1):

$$y(t) = \frac{1}{25}(-3\sin(2t) - 4\cos(2t)) + C_1e^{-t} + C_2te^{-t}$$