Review

Ex $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y^2 = t, \quad y(1)=0, \quad y'(1) = -1$

① Rewrite as first order system, with initial condition

② Use Euler's method w/ $\Delta t = 0.5$ to approximate $y(2)$

$$v = \frac{dy}{dt}$$

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -2v - y^2 + t \end{cases} \qquad \begin{cases} y(1) = 0 \leftarrow y_0 \\ v(1) = -1 \leftarrow v_0 \end{cases}$$

$$\vec{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$$

$$\vec{Y}_{k+1} = \vec{Y}_k + \Delta t \cdot \vec{F}(t_k, \vec{Y}_k)$$

$$\vec{F}(t, \vec{Y}) = \begin{pmatrix} v \\ -2v - y^2 + t \end{pmatrix}$$

$t_0 = 1 \qquad t_1 = 1.5 \qquad t_2 = 2 \qquad$ (need 2 iterations)

$$\begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} + \Delta t \begin{pmatrix} v_0 \\ -2v_0 - y_0^2 + t_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 0.5 \begin{pmatrix} -1 \\ -2\times(-1) - 0^2 + 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}$$

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} + \Delta t \begin{pmatrix} v_1 \\ -2v_1 - y_1^2 + t_1 \end{pmatrix}$$

$$= \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix} + 0.5 \begin{pmatrix} 0.5 \\ \dots \end{pmatrix} = \begin{pmatrix} \boxed{-0.25} \\ \dots \end{pmatrix}$$

Ex $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = 0$

convert to first order system, sketch phase portrait.

$$v = \frac{dy}{dt}$$

$$\begin{cases} \frac{dy}{dt} = 0 \cdot y + 1 \cdot v \\ \frac{dv}{dt} = 4y - 3v \end{cases}$$

$\vec{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$ $\quad \frac{d\vec{Y}}{dt} = A\vec{Y}$

$$A = \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}$$

$$(0-\lambda)(-3-\lambda) - 1 \cdot 4 = 0$$

$$\lambda^2 + 3\lambda - 4 = 0$$

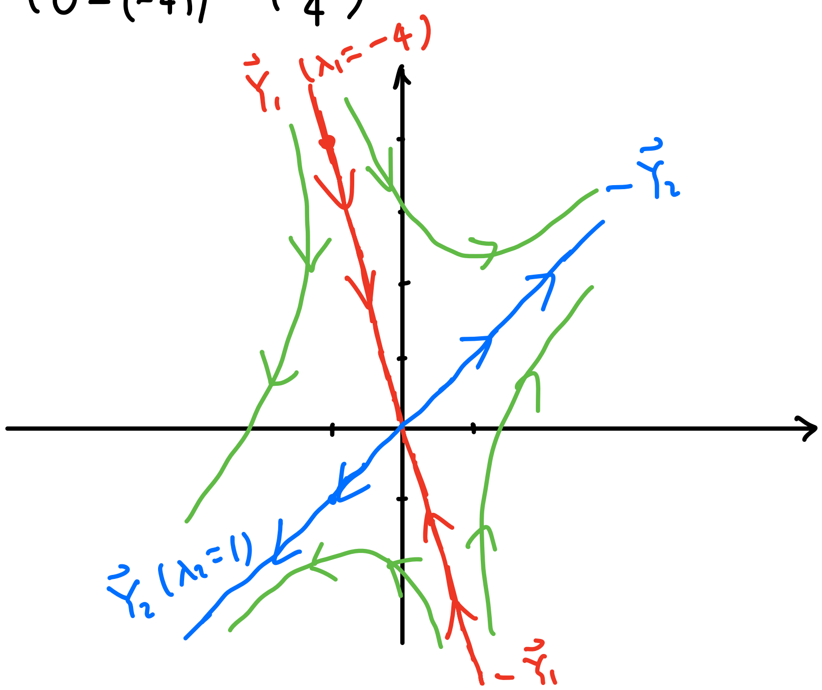
$$(\lambda+4)(\lambda-1) = 0$$

$\vec{v} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$

$\lambda_1 = -4$, $\lambda_2 = 1$ (saddle)

$\vec{v}_1 = \begin{pmatrix} -1 \\ 0-(-4) \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} -1 \\ 0-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$



Ex $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos(3t)$ ---- (1)

general sol'n, qualitative behavior of a sol'n as $t \to \infty$ (specify amplitude when applicable).

$$s^2 + 4s + 3 = 0$$

$$(s+1)(s+3) = 0$$

$$s_1 = -1, \quad s_2 = -3$$

$$\Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{-3t}$$

To find a particular sol'n to (1), Consider

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{3it} \quad \text{----- (2)}$$

Try $y(t) = Ce^{3it}$ for (2)

$$\frac{dy}{dt} = 3iCe^{3it}$$

$$\frac{d^2y}{dt^2} = -9Ce^{3it}$$

$$-9Ce^{3it} + 4 \cdot 3iCe^{3it} + 3Ce^{3it} = e^{3it}$$

$$-9C + 12iC + 3C = 1$$

$$C(-9 + 12i + 3) = 1$$

$$C = \frac{1}{-6+12i} = \frac{-6-12i}{(-6+12i)(-6-12i)}$$

$$= \frac{-6-12i}{180} = \frac{-1-2i}{30}$$

$\Rightarrow$ particular sol'n to (2) :

$$y(t) = \frac{1}{30}(-1-2i)\,e^{3it}$$

$$= \frac{1}{30}(-1-2i)(\cos(3t) + i\sin(3t))$$

$$= \frac{1}{30}\left(\underline{-\cos(3t)} - i\sin(3t) - 2i\cos(3t) \underline{+ 2\sin(3t)}\right)$$

Take real part, get particular sol'n to (1):

$$y_p(t) = \frac{1}{30}\left(-\cos(3t) + 2\sin(3t)\right)$$

$\Rightarrow$ general sol'n to (1):

$$y(t) = \underbrace{\frac{1}{30}\left(-\cos(3t) + 2\sin(3t)\right)}_{\text{oscillation}} + \underbrace{C_1 e^{-t} + C_2 e^{-3t}}_{\text{goes to } 0}$$

Qualitative behavior as $t \to \infty$

$$\text{amplitude} = \sqrt{\left(-\frac{1}{30}\right)^2 + \left(\frac{2}{30}\right)^2} = \sqrt{\frac{1+4}{30^2}} = \frac{\sqrt{5}}{30}$$

<u>Ex</u> $\dfrac{d^2y}{dt^2} + 4y = -2\sin(2t)$ ...... (1)

general sol'n, qualitative behavior of a sol'n as $t \to \infty$

$$s^2 + 4 = 0$$

$$s^2 = -4 \qquad s = \pm 2i$$

$$\Rightarrow \quad y_h(t) = C_1\cos(2t) + C_2\sin(2t)$$

To find a particular sol'n to (1), consider

$$\frac{d^2y}{dt^2} + 4y = -2e^{2it} \quad \text{------} \quad (2)$$

Try $y(t) = C\,t\,e^{2it}$ for (2)

$$\frac{dy}{dt} = C\left(e^{2it} + 2i\,t\,e^{2it}\right)$$

$$\frac{d^2y}{dt^2} = C\left(2i e^{2it} + 2i e^{2it} - 4t e^{2it}\right)$$

$$= C\left(4i e^{2it} - 4t e^{2it}\right)$$

$$C\left(4i e^{2it} - 4t e^{2it}\right) + 4Ct e^{2it} = -2e^{2it}$$

$$C\left(4i - 4t\right) + 4Ct = -2$$

$$C\left(4i - 4t + 4t\right) = -2$$

$$C = -\frac{2}{4i} = -\frac{1}{2}\cdot\frac{1}{i} = \frac{1}{2}i$$

$\Rightarrow$ particular sol'n to (2) :

$$y(t) = \frac{1}{2} i t e^{2it}$$

$$= \frac{1}{2} t\, i\left(\cos(2t) + i\sin(2t)\right)$$

$$= \frac{1}{2} t\left(i\cos(2t) - \sin(2t)\right)$$

Take imaginary part, set particular sol'n to (1) :

$$y_p(t) = \frac{1}{2} t\cos(2t)$$

$\Rightarrow$ general sol'n to (1)

$$y(t) = \underbrace{\frac{1}{2} t\cos(2t)}_{\text{oscillation w/ increasing amplitude}} + \underbrace{C_1\cos(2t) + C_2\sin(2t)}_{\text{oscillation}}$$

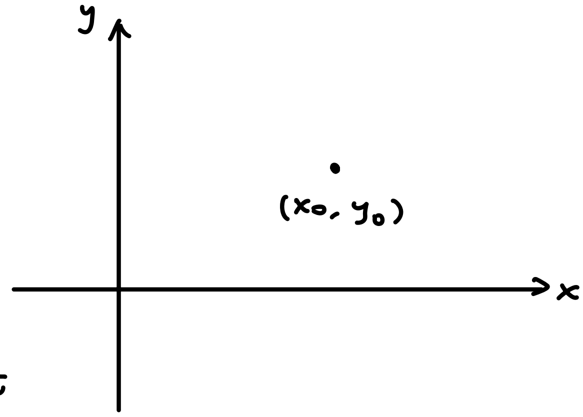
Qualitative behavior as $t \to \infty$

oscillation w/ increasing amplitude

resonance

Linearization of systems (NOT IN EXAM)

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases}$$



Assume (x_0, y_0) is an equilibrium pt

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

Near (x_0, y_0), use linearization of f and g.

$$f(x,y) \approx \cancel{f(x_0, y_0)} + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$g(x,y) \approx \cancel{g(x_0, y_0)} + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0)$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} \approx f_x(x_0,y_0)\underbrace{(x-x_0)}_{\tilde{x}(t)} + f_y(x_0,y_0)\underbrace{(y-y_0)}_{\tilde{y}(t)} \\[3ex] \dfrac{dy}{dt} \approx g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0) \end{cases}$$

$$\begin{cases} \dfrac{d\tilde{x}}{dt} \approx f_x(x_0,y_0)\,\tilde{x} + f_y(x_0,y_0)\,\tilde{y} \\[2ex] \dfrac{d\tilde{y}}{dt} \approx g_x(x_0,y_0)\,\tilde{x} + g_y(x_0,y_0)\,\tilde{y} \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \qquad A = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix}$$

Ex
$$\begin{cases} \dfrac{dx}{dt} = \underbrace{x - 2xy}_{f} \\[2ex] \dfrac{dy}{dt} = \underbrace{-2y + 3xy}_{g} \end{cases}$$

$\left(\frac{2}{3}, \frac{1}{2}\right)$ is an equilibrium pt.

$f_x = 1 - 2y \qquad f_y = -2x$

$g_x = 3y \qquad g_y = -2 + 3x$

evaluate at $\left(\frac{2}{3}, \frac{1}{2}\right)$

$$A = \begin{pmatrix} 0 & -\frac{4}{3} \\ \frac{3}{2} & 0 \end{pmatrix}$$

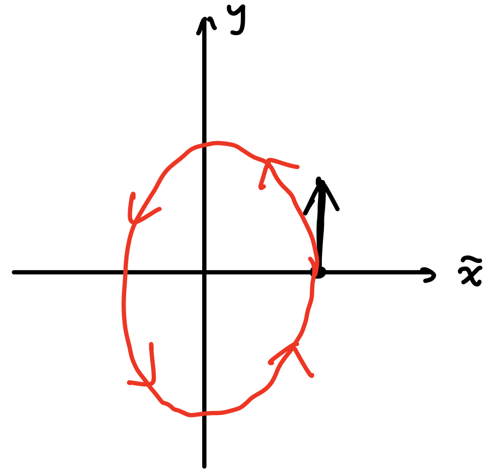
To sketch phase portrait for $\dfrac{d\tilde{Y}}{dt} = A\tilde{Y}$

$$(0-\lambda)(0-\lambda) - \left(-\frac{4}{3}\right)*\left(\frac{3}{2}\right) = 0$$

$$\lambda^2 + 2 = 0$$

$$\lambda^2 = -2$$

$$\lambda = \pm\sqrt{2}\, i \qquad \text{(center)}$$



$$\begin{pmatrix} 0 & -\frac{4}{3} \\ \frac{3}{2} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$