Review

Ex A vat has capacity 6 gallons, initially full of pure water. Salty water w/ concentration 2 oz/gallon flows in at a rate of 2 gal/min, and well-mixed liquid flows out at a rate of 3 gal/min. What is the concentration of salt in the vat when it contains 2 gallon liquid?

$V(t) = 2$ $\quad t = 4$

$$V(t) = 6 - t$$

$S(t)$: amount of salt $\qquad$ Concentration : $\dfrac{S}{6-t}$

$$\frac{dS}{dt} = 2 \times 2 - \frac{S}{6-t} \times 3 = -\frac{3}{6-t} S + 4 \qquad S(0) = 0$$

$$\frac{dS}{dt} + \underbrace{\frac{3}{6-t}}_{g(t) = \frac{3}{6-t}} S = 4$$

$$\mu(t) = e^{\int \frac{3}{6-t} dt} = e^{-3\ln(6-t)} = e^{\ln\left((6-t)^{-3}\right)} = (6-t)^{-3}$$

$$\frac{d}{dt}\left((6-t)^{-3} S\right) = 4(6-t)^{-3}$$

$$(6-t)^{-3} S = \int 4(6-t)^{-3} dt$$

$$= 2(6-t)^{-2} + C$$

$$\int 4(6-t)^{-3} dt \qquad u = 6-t$$

$$= -4\int u^{-3} du \qquad du = -dt$$

$$= -4 \cdot \frac{1}{-2} u^{-2} + C$$

$$= 2(6-t)^{-2} + C$$

initial condition $\rightsquigarrow$

$$0 = 2 \cdot 6^{-2} + C \qquad C = -\frac{1}{18}$$

$$(6-t)^{-3} S = 2(6-t)^{-2} - \frac{1}{18}$$

$$S = 2(6-t) - \frac{1}{18}(6-t)^3$$

concentration at $t=4$: $\frac{S(4)}{2} = \frac{2 \times 2 - \frac{1}{18} \times 2^3}{2} = \frac{16}{9}$

Ex
$$\begin{cases} \frac{dy_1}{dt} = -y_1 - y_2 \\ \frac{dy_2}{dt} = 3y_1 - 5y_2 \end{cases}$$

phase portrait.

$$A = \begin{pmatrix} -1 & -1 \\ 3 & -5 \end{pmatrix}$$

$$(-1-\lambda)(-5-\lambda) - (-1) \times 3 = 0$$

$$\lambda^2 + 6\lambda + 5 + 3 = 0$$

$$\lambda^2 + 6\lambda + 8 = 0$$

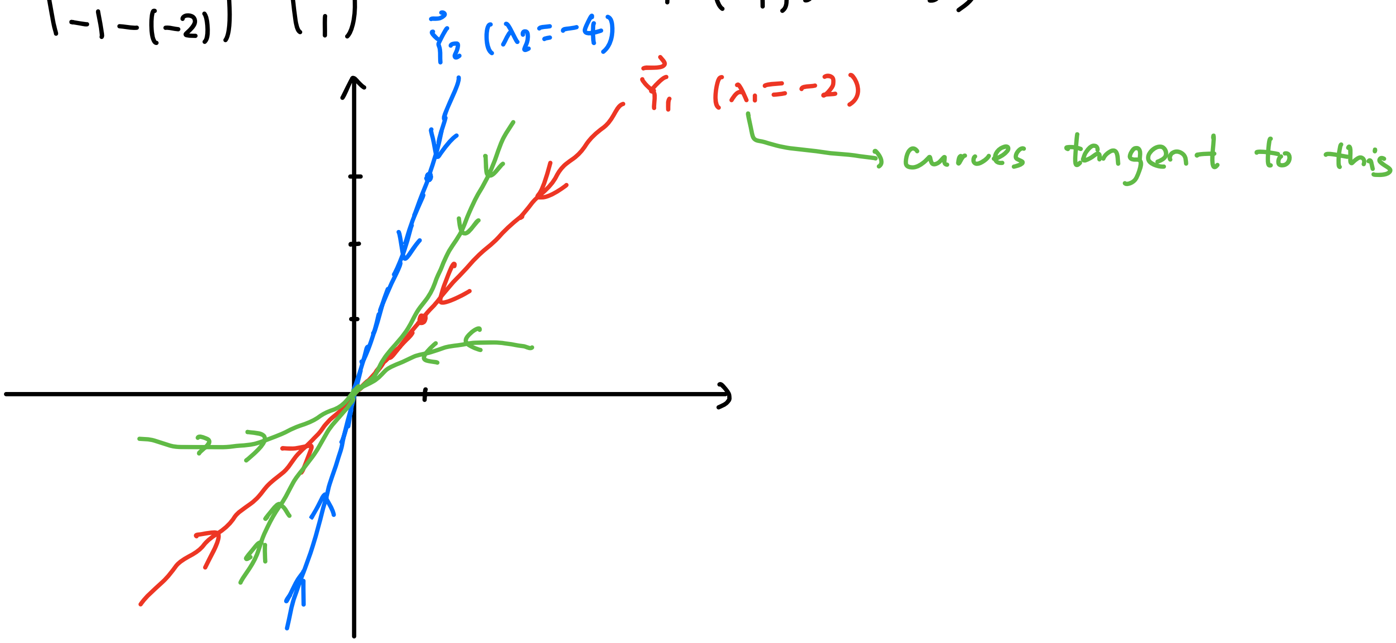
$$(\lambda+2)(\lambda+4) = 0$$

$\lambda_1 = -2$, $\lambda_2 = -4$ (sink)

$$\vec{v} = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1-(-4) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

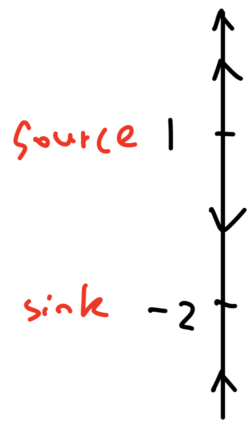


Ex $\dfrac{dy}{dt} = \underbrace{y^2}_{a=1} + \underbrace{\mu y}_{b=\mu} \underbrace{- \mu^2 - 1}_{c=-\mu^2-1}$, where μ is a parameter

① Sketch phase line when $\mu = 1$, type of equilibrium pts

② Find bifurcation values

① $\dfrac{dy}{dt} = y^2 + y - 2 = f(y)$



$f(2) = 2^2 + 2 - 2 = 4$

$f(0) = -2$

$f(-3) = (-3)^2 - 3 - 2$
$= 4$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y_1 = -2, \; y_2 = 1$

② $b^2 - 4ac = \mu^2 - 4 \cdot 1 \cdot (-\mu^2 - 1) = 0$

$\mu^2 + 4\mu^2 + 4 = 0$

$5\mu^2 + 4 = 0$ no sol'n

$\Rightarrow$ no bifurcation values.

Ex $\dfrac{dy}{dt} = -y^3(2t+1)$, $y(0) = 2$

Solve, find domain of definition

$$\int \frac{dy}{y^3} = -\int (2t+1)\,dt$$

$$\frac{1}{-2} \cdot \frac{1}{y^2} = -(t^2 + t) + C$$

initial condition $\rightsquigarrow$ $\dfrac{1}{-2} \cdot \dfrac{1}{2^2} = C$ $\qquad C = -\dfrac{1}{8}$

$$-\frac{1}{2} \cdot \frac{1}{y^2} = -(t^2 + t) - \frac{1}{8}$$

$$\frac{1}{y^2} = 2t^2 + 2t + \frac{1}{4}$$

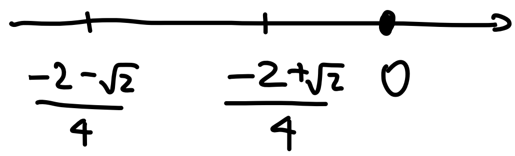
$$y^2 = \frac{1}{2t^2 + 2t + \frac{1}{4}}$$

$$y = \pm \frac{1}{\sqrt{2t^2 + 2t + \frac{1}{4}}} = \frac{1}{\sqrt{2t^2 + 2t + \frac{1}{4}}}$$

choose "+" because $y(0) = 2$

Domain of definition:

$$2t^2 + 2t + \frac{1}{4} = 0$$



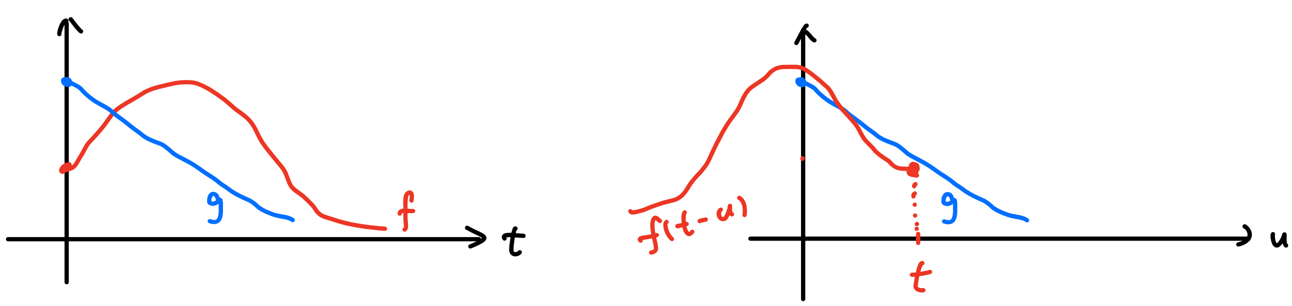
$$t = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot \frac{1}{4}}}{2 \times 2} = \frac{-2 \pm \sqrt{2}}{4}$$

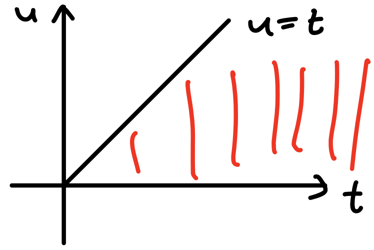
$$\Rightarrow \left(\frac{-2+\sqrt{2}}{4}, \infty \right)$$

6.5 Convolutions (NOT in exam)

Def Let $f(t)$, $g(t)$ be functions on $[0, \infty)$. The convolution of f and g is a function $f * g$ on $[0, \infty)$ defined by

$$(f * g)(t) = \int_0^t f(t-u)\, g(u)\, du$$





$$\mathcal{L}[f*g] = \int_0^\infty (f*g)(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \int_0^t f(t-u)\, g(u)\, du\; e^{-st}\, dt$$

$$= \int_0^\infty \int_u^\infty f(t-u)\, e^{-st}\, dt\; g(u)\, du$$

$v = t-u$

$dv = dt$

$$= \int_0^\infty \int_0^\infty f(v)\, \underbrace{e^{-s(u+v)}}_{e^{-su-sv}}\, dv\; g(u)\, du$$

$$= \int_0^\infty \int_0^\infty f(v)\, e^{-sv}\, dv\; e^{-su}\, g(u)\, du$$

$$= \mathcal{L}[f]\cdot\mathcal{L}[g]$$

Consider $\dfrac{d^2y}{dt^2} + p\dfrac{dy}{dt} + qy = g(t)$, $y(0)=0$, $y'(0)=0$

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + p\,\mathcal{L}\left[\frac{dy}{dt}\right] + q\,\mathcal{L}[y] = \mathcal{L}[g]$$

$$s^2\mathcal{L}[y] - y'(0) - s\,y(0) + p\left(s\mathcal{L}[y] - y(0)\right) + q\,\mathcal{L}[y] = \mathcal{L}[g]$$

$$(s^2+ps+q)\,\mathcal{L}[y] = \mathcal{L}[g]$$

$$\mathcal{L}[y] = \underbrace{\frac{1}{s^2+ps+q}}_{\mathcal{L}[f]}\,\mathcal{L}[g] \qquad f = \mathcal{L}^{-1}\left[\frac{1}{s^2+ps+q}\right]$$

$$\Rightarrow\quad y(t) = (f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

- If one only knows some point values of g, this helps us give an approximation of $y(t)$.