Total time: 75 minutes.
Total points: 100.

Problem 1 (10 points). Use Euler's method with $\Delta t = 0.5$ to approximate the solution to the following system at $t = 3$.

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = ty \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2x \end{cases}, \qquad \begin{cases} x(2) = 1 \\ y(2) = 0 \end{cases}$$

$$t_0 = 2, \quad t_1 = 2.5, \quad t_2 = 3$$

need 2 iterations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \Delta t \begin{pmatrix} t_0 y_0 \\ -2x_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0.5 \begin{pmatrix} 2 \times 0 \\ -2 \times 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \Delta t \begin{pmatrix} t_1 y_1 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0.5 \begin{pmatrix} 2.5 \times (-1) \\ -2 \times 1 \end{pmatrix} = \begin{pmatrix} -0.25 \\ -2 \end{pmatrix}$$

Problem 2 (20 points). Consider the following first order system:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}$$

Find the <u>general solution</u> and sketch the <u>phase portrait</u>. Specify the <u>type</u> of the phase portrait.

Eigenvalues:

$$(-3-\lambda)(-1-\lambda) - (-1) \times 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2} = -2 \pm i$$

Type of phase portrait: spiral sink.
To sketch phase portrait, calculate vector field at $(1, 0)$:

$$\begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

Then we find general solution. Take the eigenvalue $\lambda = -2 + i$, eigenvector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -3 - (-2 + i) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 - i \end{pmatrix}$$

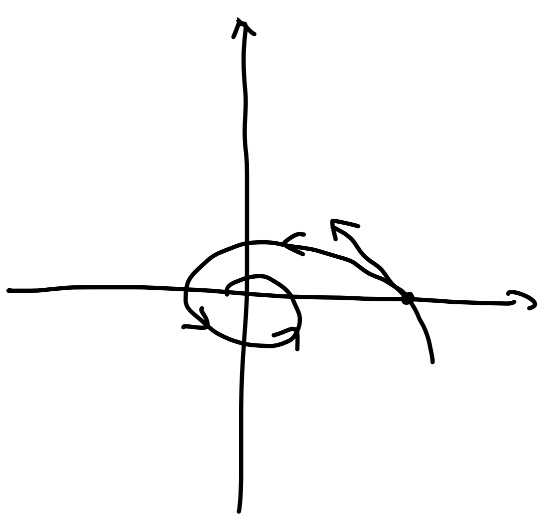
Complex solution

$$\mathbf{Y} = e^{(-2+i)t} \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} = e^{-2t}(\cos t + i \sin t) \begin{pmatrix} 1 \\ -1 - i \end{pmatrix} = e^{-2t} \begin{pmatrix} \cos t + i \sin t \\ -\cos t - i \sin t - i \cos t + \sin t \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + i e^{-2t} \begin{pmatrix} \sin t \\ -\sin t - \cos t \end{pmatrix}$$

Therefore the general solution is

$$\mathbf{Y} = C_1 e^{-2t} \begin{pmatrix} \cos t \\ -\cos t + \sin t \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} \sin t \\ -\sin t - \cos t \end{pmatrix}$$



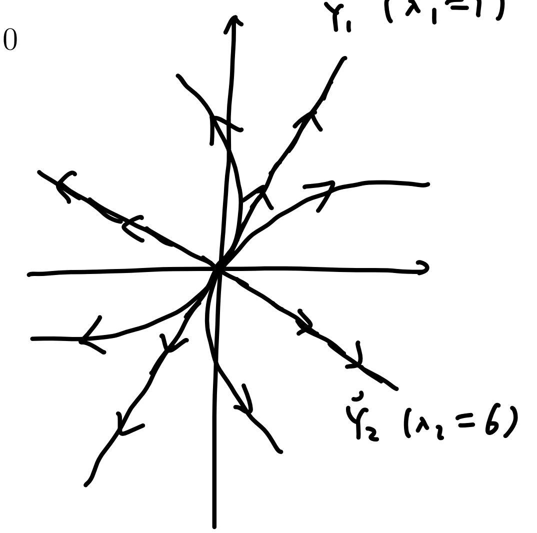
Problem 3 ($25 = 20 + 5$ points). Consider the following first order system:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \mathbf{Y}$$

(1) Find the general solution and sketch the phase portrait. Specify the type of the phase portrait.

Eigenvalues:

$$(5-\lambda)(2-\lambda) - (-2)\times(-2) = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda - 1)(\lambda - 6) = 0$$
$$\lambda_1 = 1, \quad \lambda_2 = 6$$



Type of phase portrait: source.
For $\lambda_1 = 1$, eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

For $\lambda_2 = 6$, eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 5-6 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

General solution

$$\mathbf{Y} = C_1 e^t \begin{pmatrix} 2 \\ 4 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

To sketch phase portrait, first sketch the two straight-line solutions. Then the curves are tangent to the line corresponding to $\lambda_1 = 1$ (which has smaller absolute value).

(2) Solve the initial value problem with initial condition

$$\mathbf{Y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Plugging $t = 0$ into general solution,

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{cases} 3 = 2C_1 + 2C_2 \\ 1 = 4C_1 - C_2 \end{cases}$$

Solve to get

$$C_1 = \frac{1}{2}, \quad C_2 = 1$$

Therefore the solution is

$$\mathbf{Y} = \frac{1}{2} e^t \begin{pmatrix} 2 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Problem 4 (20 points). Find the general solution to

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 3e^{-2t}$$

Eigenvalues:

$$s^2 + 6s + 8 = 0$$

$$(s+2)(s+4) = 0$$

$$s_1 = -2, \quad s_2 = -4$$

$$y_h = C_1 e^{-2t} + C_2 e^{-4t}$$

To find a particular solution, try

$$y = Cte^{-2t}$$

(because -2 is a single eigenvalue).

$$\frac{dy}{dt} = C(e^{-2t} - 2te^{-2t}), \quad \frac{d^2y}{dt^2} = C(-4e^{-2t} + 4te^{-2t})$$

Plugging into the DE,

$$C(-4e^{-2t} + 4te^{-2t}) + 6C(e^{-2t} - 2te^{-2t}) + 8Cte^{-2t} = 3e^{-2t}$$

$$C(-4 + 4t) + 6C(1 - 2t) + 8Ct = 3$$

$$C(-4 + 4t + 6 - 12t + 8t) = 3$$

$$C = \frac{3}{2}$$

Therefore a particular solution is

$$y_p = \frac{3}{2}te^{-2t}$$

General solution is

$$y = \frac{3}{2}te^{-2t} + C_1 e^{-2t} + C_2 e^{-4t}$$

Problem 5 ($20+5=25$ points). Consider the following differential equation.

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+6\frac{\mathrm{d}y}{\mathrm{d}t}+10y=\sin(2t)$$

(1) Find the general solution.

Eigenvalues:

$$s^2+6s+10=0$$
$$s=\frac{-6\pm\sqrt{6^2-4\times 1\times 10}}{2}=-3\pm i$$
$$y_h=C_1e^{-3t}\cos t+C_2e^{-3t}\sin t$$

Call the original DE as (1). To find a particular solution to (1), consider the following DE (call it (2))

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+6\frac{\mathrm{d}y}{\mathrm{d}t}+10y=e^{2it}$$

Try

$$y=Ce^{2it}$$

for (2).

$$\frac{\mathrm{d}y}{\mathrm{d}t}=2iCe^{2it},\quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2}=-4Ce^{2it}$$
$$-4Ce^{2it}+6\times 2iCe^{2it}+10Ce^{2it}=e^{2it}$$
$$-4C+12iC+10C=1$$
$$(6+12i)C=1$$
$$C=\frac{1}{6+12i}=\frac{1}{6}\cdot\frac{1}{1+2i}=\frac{1}{6}\cdot\frac{1-2i}{5}=\frac{1}{30}(1-2i)$$

Get particular solution to (2):

$$y=\frac{1}{30}(1-2i)e^{2it}=\frac{1}{30}(1-2i)(\cos(2t)+i\sin(2t))=\frac{1}{30}(\cos(2t)+i\sin(2t)-2i\cos(2t)+2\sin(2t))$$

Take imaginary part, get particular solution to (1):

$$y_p=\frac{1}{30}(\sin(2t)-2\cos(2t))$$

General solution to (1):

$$y=\frac{1}{30}(\sin(2t)-2\cos(2t))+C_1e^{-3t}\cos t+C_2e^{-3t}\sin t$$

(2) Describe the qualitative behavior of a solution to this differential equation as t gets large. Specify the amplitude if applicable.

Oscillates with a fixed amplitude. The amplitude is

$$A=\sqrt{(\frac{1}{30})^2+(\frac{-2}{30})^2}=\frac{\sqrt{5}}{30}$$