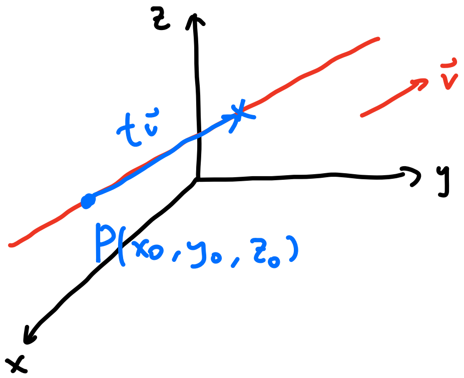
2.5 Equation of lines and planes in space

- Equation for a line



- A direction vector $\vec{v}$ of a line is a nonzero vector that is parallel to the line
- If we know $\vec{v}$ and a pt $P(x_0, y_0, z_0)$ on the line, then the vector equation for the line is

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

where $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

- In components, we get parametric equation for the line

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \qquad \vec{v} = \langle a, b, c \rangle$$

Ex Find vector equation, parametric equation for the line passing $P(1, -1, 2)$ and $Q(0, -3, 1)$

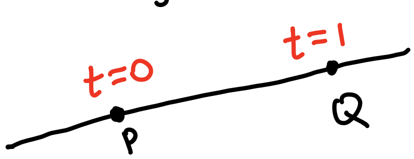
$$\vec{v} = \overrightarrow{PQ} = \langle -1, -2, -1 \rangle$$

$\Rightarrow$ vector equation : $\vec{r} = \langle 1, -1, 2 \rangle + t\langle -1, -2, -1 \rangle$

if we want segment PQ, then take $0 \le t \le 1$.

parametric equation : $\begin{cases} x = 1 - t \\ y = -1 - 2t \\ z = 2 - t \end{cases}$

- The segment connecting P, Q

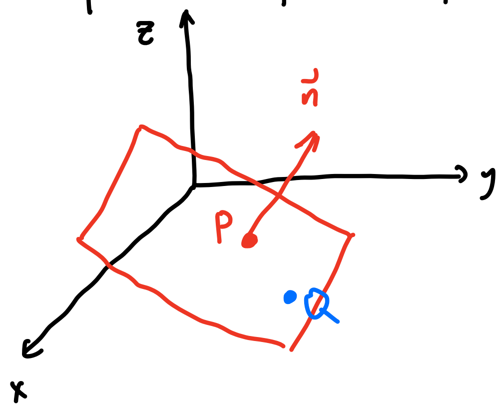


$\vec{r} = \vec{r}_0 + t\vec{v}$ $\qquad \vec{v} = \overrightarrow{PQ}$

where $\vec{r}_0$ is position vector of P

$\Rightarrow$ segment PQ corresponds $0 \le t \le 1$

- Equation for a plane



- A <u>normal vector</u> $\vec{n}$ of a plane is a nonzero vector that is perpendicular to the plane.
- If we know $\vec{n}$ and a pt $P(x_0, y_0, z_0)$ on the plane, then the equation for the plane is

$$\vec{n} \cdot \vec{PQ} = 0 \quad \text{where } Q(x, y, z)$$

- Write $\vec{n} = \langle a, b, c \rangle$. Then

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

<u>Ex</u> Find the equation for the plane containing $P(0, 1, 2)$, $Q(1, 2, -1)$, $R(-2, 1, 3)$

$\vec{PQ} = \langle 1, 1, -3 \rangle \qquad \vec{PR} = \langle -2, 0, 1 \rangle$

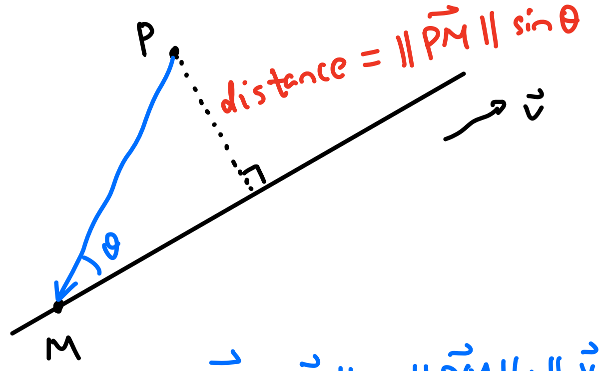
$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -2 & 0 & 1 \end{vmatrix} = \langle 1, 5, 2 \rangle$$

$\Rightarrow$ equation for the plane $\qquad x + 5y + 2z - 9 = 0$

$$1 \cdot (x - 0) + 5 \cdot (y - 1) + 2(z - 2) = 0$$

$$\left(\text{or } 1 \cdot (x - 1) + 5 \cdot (y - 2) + 2(z + 1) = 0 \right)$$

- Distance: pt to line



$$\text{distance} = \frac{\|\vec{PM} \times \vec{v}\|}{\|\vec{v}\|}$$

recall: $\|\vec{PM} \times \vec{v}\| = \|\vec{PM}\| \cdot \|\vec{v}\| \cdot \sin\theta$

__Ex__ Find distance from $P(3,2,1)$ to the line

$$\vec{r} = \langle 2t, 1+t, -1-t \rangle$$

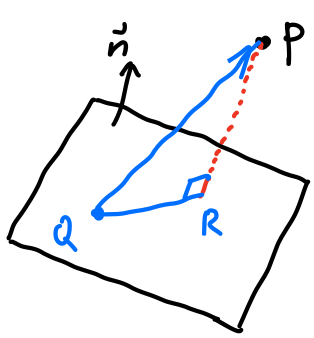
$\vec{v} = \langle 2, 1, -1 \rangle$ $\qquad$ $M(0,1,-1)$

$\vec{PM} = \langle -3, -1, -2 \rangle$

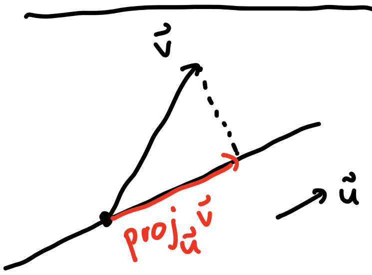
$$\vec{PM} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -1 & -2 \\ 2 & 1 & -1 \end{vmatrix} = \langle 3, -7, -1 \rangle$$

$$\text{distance} = \frac{\sqrt{3^2 + (-7)^2 + (-1)^2}}{\sqrt{2^2 + 1^2 + (-1)^2}} = \frac{\sqrt{59}}{\sqrt{6}}$$

- Distance: pt to plane



$$\text{distance} = \left| \text{comp}_{\vec{n}} \vec{QP} \right| = \frac{|\vec{QP} \cdot \vec{n}|}{\|\vec{n}\|}$$



$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}}_{\text{comp}_{\vec{u}} \vec{v}} \frac{\vec{u}}{\|\vec{u}\|}$$

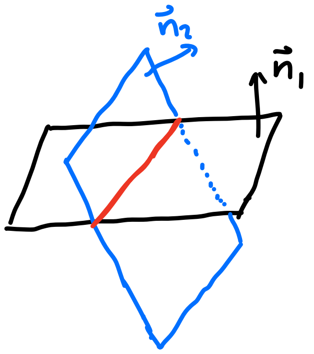
Ex Find distance from $P(1,-1,0)$ to the plane $2x+y-z=1$

$\vec{n} = \langle 2, 1, -1 \rangle$, $Q(0,0,-1)$

$\vec{QP} = \langle 1, -1, 1 \rangle$

$$\text{distance} = \frac{|2+(-1)+(-1)|}{\sqrt{2^2+1^2+(-1)^2}} = 0$$

- Line of intersection between planes



- direction vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$
- find a pt on BOTH planes

$\rightsquigarrow$ vector equation

Ex Line of intersection between

$x+2y+z=1$ and $2x-y+z=0$

$\vec{n}_1 = \langle 1, 2, 1 \rangle$ $\quad$ $\vec{n}_2 = \langle 2, -1, 1 \rangle$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \langle 3, 1, -5 \rangle$$

To find a pt on both planes: set $z=0 \Rightarrow \begin{cases} x+2y=1 \\ 2x-y=0 \end{cases}$ $\quad y=2x$

$x+4x=1 \quad 5x=1 \quad x=\frac{1}{5} \quad y=\frac{2}{5}$

$\Rightarrow \left(\frac{1}{5}, \frac{2}{5}, 0\right)$

$\Rightarrow$ line of intersection: $\vec{r} = \langle \frac{1}{5}, \frac{2}{5}, 0 \rangle + t\langle 3, 1, -5 \rangle$

- Angle between planes

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|} \qquad \theta = \cos^{-1}\left(\frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \cdot \|\vec{n}_2\|}\right)$$

Ex Angle between:

$$x + 2y + z = 1 \quad \text{and} \quad 2x - y + z = 0$$

$$\vec{n}_1 = \langle 1, 2, 1 \rangle \qquad \vec{n}_2 = \langle 2, -1, 1 \rangle$$

$$\vec{n}_1 \cdot \vec{n}_2 = 2 + (-2) + 1 = 1$$

$$\|\vec{n}_1\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \qquad \|\vec{n}_2\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\theta = \cos^{-1}\left(\frac{|1|}{\sqrt{6} \cdot \sqrt{6}} \right) = \cos^{-1}\left(\frac{1}{6}\right).$$