3.1, 3.2 Vector-valued functions

Def A vector-valued function is a function of the form

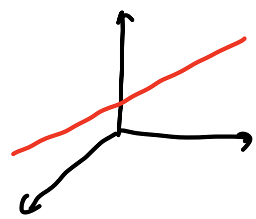
$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} = \langle f(t), g(t), h(t) \rangle$$

where the component functions f, g, h are real valued functions

- A vector-valued function represents a curve in space.

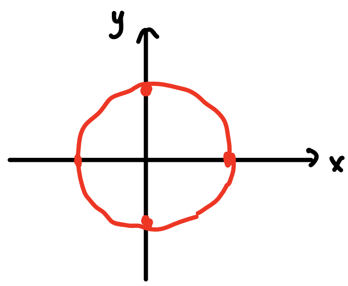
For example,

$\vec{r}(t) = \langle 2-t, 3+t, 3+2t \rangle$ represents a line

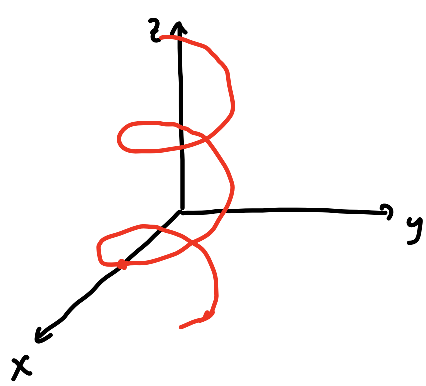


$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$ (in two dimensions)

represents the unit circle



$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ represents a helix



- Limit, derivative, integral (definite/indefinite) for vector-valued functions:

Do those on each component!

Ex $\lim_{t\to 1} \langle t-1,\ 2t-1,\ \frac{t^2-1}{t-1} \rangle = \langle 0, 1, 2 \rangle$

$\hookrightarrow t+1$

$$\frac{d}{dt}\langle t^2, \sin t, te^{-t} \rangle = \langle 2t, \cos t, e^{-t} + t\cdot(-e^{-t}) \rangle$$

$$\int \langle t^2, \sin t \rangle dt = \langle \tfrac{1}{3}t^3 + C_1,\ -\cos t + C_2 \rangle$$

$$= \langle \tfrac{1}{3}t^3, -\cos t \rangle + \vec{C} \qquad (\text{where } \vec{C} = \langle C_1, C_2 \rangle)$$

$$\int_0^1 \langle t^2, \sin t \rangle dt = \langle \tfrac{1}{3}t^3 \Big|_{t=0}^{1},\ -\cos t \Big|_{t=0}^{1} \rangle$$

$$= \langle \tfrac{1}{3},\ -\cos 1 + 1 \rangle$$

- Product rules for derivative

$$\frac{d}{dt}(fg) = f'g + fg'$$

$$\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{u}(t)\right) = \vec{r}'(t)\cdot\vec{u}(t) + \vec{r}(t)\cdot\vec{u}'(t)$$

$$\frac{d}{dt}\left(\vec{r}(t)\times\vec{u}(t)\right) = \vec{r}'(t)\times\vec{u}(t) + \vec{r}(t)\times\vec{u}'(t)$$

- Chain rules

$$\frac{d}{dt}\left(g(f(t))\right) = g'(f(t))\cdot f'(t)$$

$$\frac{d}{dt}\left(\vec{r}(f(t))\right) = \vec{r}'(f(t))\cdot f'(t)$$

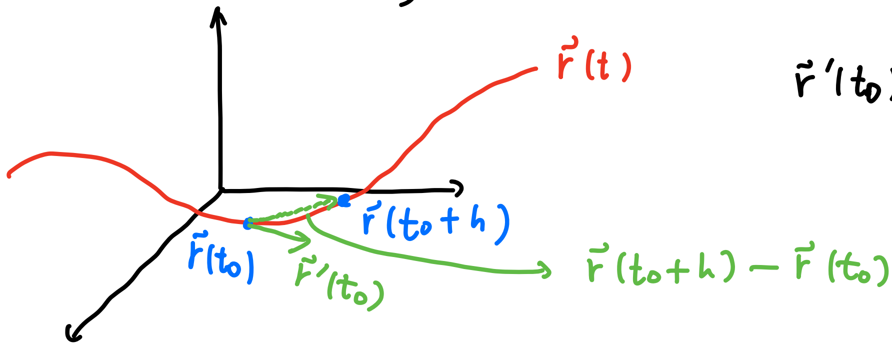
- Use in proofs : for example,

$\vec{r}'(t)\cdot\vec{r}(t) = 0$ for any t $\Leftrightarrow$ $\|\vec{r}(t)\|$ is constant in t

Proof $\frac{d}{dt}\left(\|\vec{r}(t)\|^2\right) = \frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right) = \vec{r}'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}'(t)$

$$= 2\,\vec{r}(t)\cdot\vec{r}'(t)$$

- Geometric meaning of $\vec{r}\,'(t)$: <u>tangent vector</u>



$$\vec{r}\,'(t_0) = \lim_{h\to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h}$$

<u>Def</u> The <u>unit tangent vector</u> of a curve $\vec{r}(t)$ is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

(defined for t values for which $\vec{r}\,'(t) \neq \vec{0}$)

<u>Ex</u> Find unit tangent vector for $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 3t^2\,\vec{k}$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 6t \rangle$$

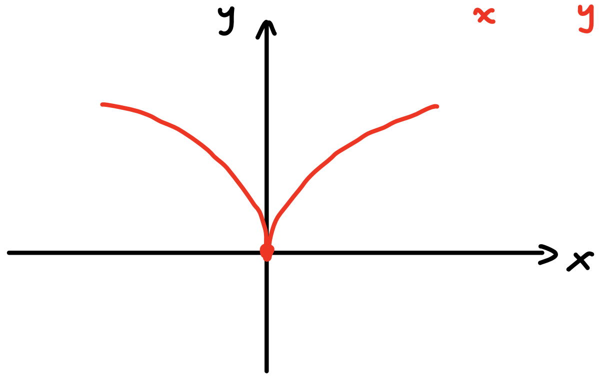
$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (6t)^2} = \sqrt{1 + 36t^2}$$

$$\vec{T}(t) = \frac{1}{\sqrt{1+36t^2}} \langle -\sin t, \cos t, 6t \rangle$$

- A curve $\vec{r}(t)$ may have singularity at pts where $\vec{r}\,'(t) = \vec{0}$

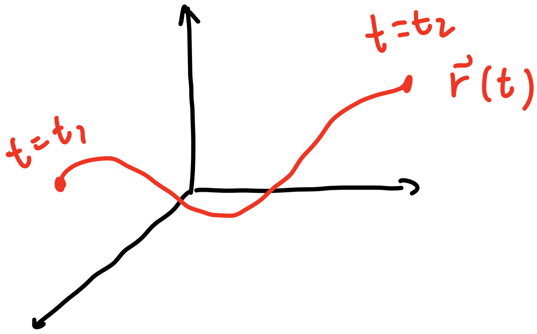
For example, $\vec{r}(t) = \langle \underset{\downarrow\, x}{t^3}, \underset{\downarrow\, y}{t^2} \rangle$ $\qquad \vec{r}\,'(t) = \langle 3t^2, 2t \rangle$

$\vec{r}\,'(t) = \vec{0}$ when $t = 0$



$y = x^{\frac{2}{3}}$

3.3 Arclength



<u>Thm</u> Arclength of a curve $\vec{r}(t)$ from $t=t_1$ to $t=t_2$ is

$$s = \int_{t_1}^{t_2} \|\vec{r}\,'(t)\|\,dt = \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2+h'(t)^2}\,dt$$

where $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

<u>Ex</u> Arclength of $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$ from $t=0$ to $t=2\pi$

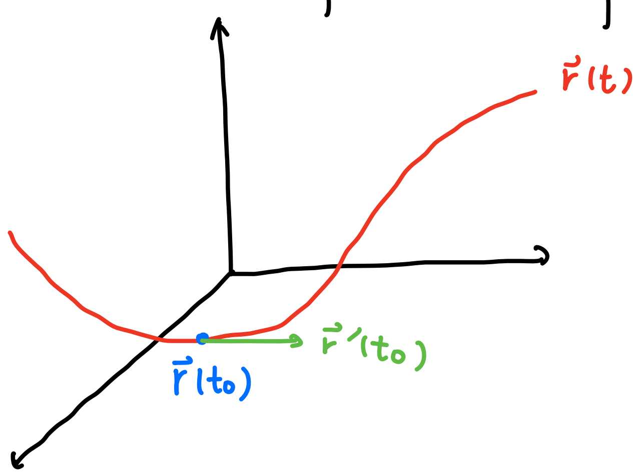
$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 2^2} = \sqrt{5}$$

$$s = \int_0^{2\pi} \sqrt{5}\,dt = \sqrt{5}\,t \Big|_{t=0}^{2\pi} = \sqrt{5}\cdot 2\pi$$

3.4 Motion in space

One can use a vector-valued function $\vec{r}(t)$ to describe the motion of a particle in space: its position at time t is $\vec{r}(t)$



- Its <u>velocity</u> is $\vec{v}(t) = \vec{r}\,'(t)$
- Its <u>speed</u> is $\|\vec{v}(t)\|$
- Its <u>acceleration</u> is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

$$\vec{r}\,'(t_0) = \lim_{h\to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h}$$

Ex The position of a moving particle is $\vec{r}(t) = \cos(3t)\,\vec{i} + 2t\,\vec{j} + \vec{k}$

Find velocity, speed, acceleration

$\vec{v}(t) = -3\sin(3t)\,\vec{i} + 2\,\vec{j}$

speed $= \sqrt{(-3\sin(3t))^2 + 2^2}$

$\vec{a}(t) = -9\cos(3t)\,\vec{i}$.