3.4 (continued)

Recall:

$\vec{r}(t)$: position

$\vec{v}(t) = \vec{r}\,'(t)$: velocity $\qquad \|\vec{v}(t)\|$: speed

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$: acceleration

Ex The acceleration of a moving particle is $\vec{a}(t) = \langle t, t^2 \rangle$. At $t=0$, the particle is at $(1,0)$ and is at rest. Find its position function $\vec{r}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle t, t^2 \rangle dt = \langle \tfrac{1}{2}t^2, \tfrac{1}{3}t^3 \rangle + \vec{C}_1$$

$$\vec{v}(0) = \vec{0} \rightsquigarrow \langle 0,0 \rangle = \langle 0,0 \rangle + \vec{C}_1 \qquad \vec{C}_1 = \vec{0}$$

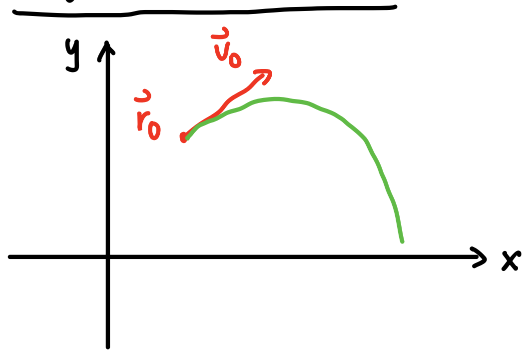
$$\Rightarrow \vec{v}(t) = \langle \tfrac{1}{2}t^2, \tfrac{1}{3}t^3 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle \tfrac{1}{2}t^2, \tfrac{1}{3}t^3 \rangle dt = \langle \tfrac{1}{6}t^3, \tfrac{1}{12}t^4 \rangle + \vec{C}_2$$

$$\vec{r}(0) = \langle 1,0 \rangle \rightsquigarrow \langle 1,0 \rangle = \langle 0,0 \rangle + \vec{C}_2 \qquad \vec{C}_2 = \langle 1,0 \rangle$$

$$\Rightarrow \vec{r}(t) = \langle \tfrac{1}{6}t^3, \tfrac{1}{12}t^4 \rangle + \langle 1,0 \rangle = \langle \tfrac{1}{6}t^3 + 1, \tfrac{1}{12}t^4 \rangle$$

Projectile motion



Newton's 2nd law: $\vec{F} = m\vec{a}$

$\vec{F}$: gravity $\qquad \vec{F} = -mg\vec{j}$

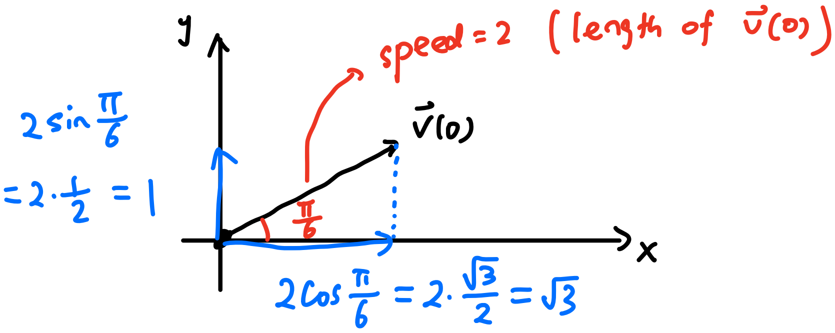
$$\Rightarrow \vec{a} = -g\vec{j}$$

Ex A projectile is launched from $(0,0)$ w/ speed 2 and an angle $\frac{\pi}{6}$ above horizon

(units: ft, sec)

- Find its position function (take $g = 32$)
- Find the time at which it falls on ground
- Find its maximal height.

$\vec{a}(t) = -g\,\vec{j} = -32\,\vec{j}$ $\quad\quad$ $\vec{r}(0) = \langle 0, 0 \rangle$ $\quad\quad$ $\vec{v}(0) = \langle \sqrt{3}, 1 \rangle$



$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, -32 \rangle\,dt = \langle 0, -32t \rangle + \vec{C_1}$$

$\vec{v}(0) = \langle \sqrt{3}, 1 \rangle \rightsquigarrow \langle \sqrt{3}, 1 \rangle = \langle 0, 0 \rangle + \vec{C_1}$ $\quad\quad$ $\vec{C_1} = \langle \sqrt{3}, 1 \rangle$

$$\Rightarrow \vec{v}(t) = \langle 0, -32t \rangle + \langle \sqrt{3}, 1 \rangle = \langle \sqrt{3}, -32t + 1 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle \sqrt{3}, -32t + 1 \rangle\,dt = \langle \sqrt{3}\,t, -16t^2 + t \rangle + \vec{C_2}$$

$\vec{r}(0) = \langle 0, 0 \rangle \rightsquigarrow \langle 0, 0 \rangle = \langle 0, 0 \rangle + \vec{C_2}$ $\quad\quad$ $\vec{C_2} = \vec{0}$

$$\Rightarrow \vec{r}(t) = \langle \sqrt{3}\,t, -16t^2 + t \rangle$$

- "Falls on ground": $\quad -16t^2 + t = 0 \quad\quad t(-16t + 1) = 0$

  $t \neq 0 \Rightarrow -16t + 1 = 1 \quad\quad t = \frac{1}{16}$

- "Maximal height": max of $f(t) = -16t^2 + t$

  $f'(t) = -32t + 1 = 0 \quad\quad t = \frac{1}{32} \quad\quad \Rightarrow$ max of f is $f(\frac{1}{32}) = -16\cdot(\frac{1}{32})^2 + \frac{1}{32}$

  $= \frac{1}{64}$.

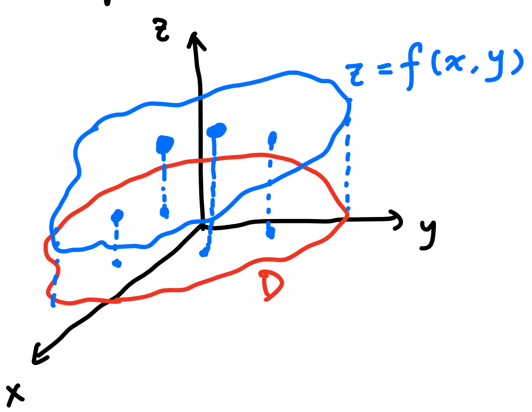
4.1 Functions of several variables

Def A function of two variables $z = f(x,y)$ maps a pair (x,y) in a subset D of $\mathbb{R}^2$ to a real number z.

D : domain of f

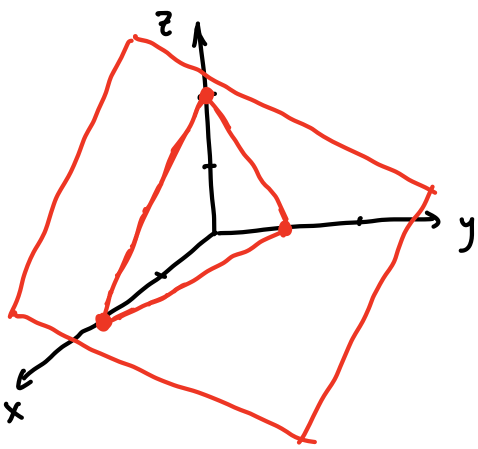
All possible outputs z : range of f

- Graph of $f(x,y)$: all pts (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$



It forms a surface.

For example, the graph of $f(x,y) = 2 - x - 2y$

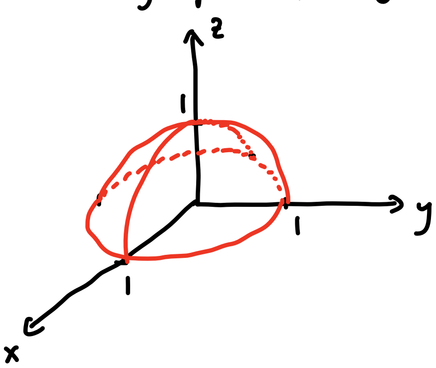


$z = 2 - x - 2y$

$x + 2y + z = 2$ (a plane)

domain: $\mathbb{R}^2$

The graph of $g(x,y) = \sqrt{1 - x^2 - y^2}$



$z = \sqrt{1 - x^2 - y^2}$

$z^2 = 1 - x^2 - y^2$

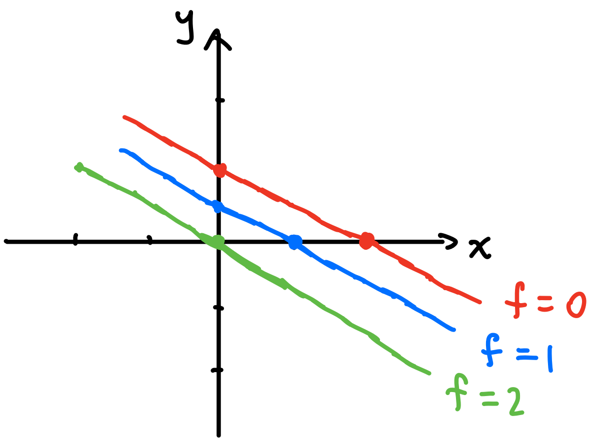
$x^2 + y^2 + z^2 = 1 \quad (z \geq 0)$

domain: $\{(x,y) : 1 - x^2 - y^2 \geq 0\}$

- Level curves of $f(x,y)$ for the value c: pts (x,y) such that $f(x,y)=c$. A contour map consists of several level curves

Examples:

$$f(x,y) = 2 - x - 2y$$

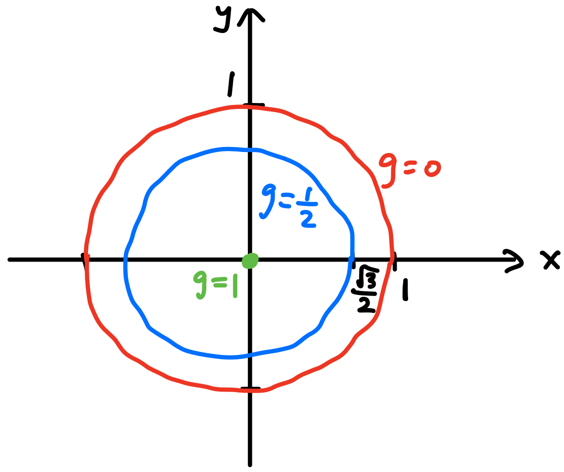


$f=0$: $\quad 2-x-2y=0$

$f=1$: $\quad 2-x-2y=1$

$f=2$: $\quad 2-x-2y=2$

$$g(x,y) = \sqrt{1-x^2-y^2}$$



$g=0$: $\quad \sqrt{1-x^2-y^2}=0 \qquad x^2+y^2=1$

$g=\frac{1}{2}$: $\quad \sqrt{1-x^2-y^2}=\frac{1}{2} \qquad x^2+y^2=\frac{3}{4}$

$g=1$: $\quad \sqrt{1-x^2-y^2}=1 \qquad x^2+y^2=0$

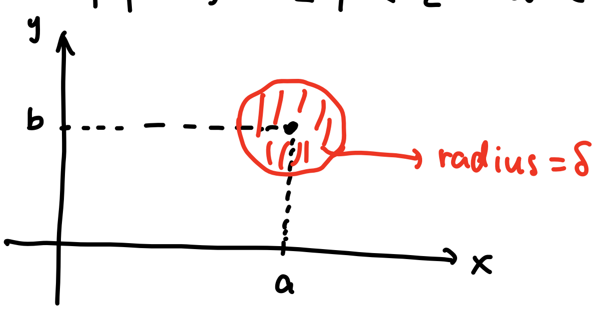
4.2 Limits and continuity

Def Let f be a function of two variables. We say

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$$



Limit laws

$$\lim_{(x,y)\to(a,b)} c = c \qquad \lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b$$

$$\lim_{(x,y)\to(a,b)} \big(f(x,y) + g(x,y)\big) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

(provided f, g limits exist)

...

$$\lim_{(x,y)\to(2,1)} xy = \lim_{(x,y)\to(2,1)} x \cdot \lim_{(x,y)\to(2,1)} y = 2 \times 1 = 2 \quad .$$