4.2 (continued)

Def $f(x,y)$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

(need both sides to exist, and equal)

"be careful when (a,b) is on the boundary of the domain of f"

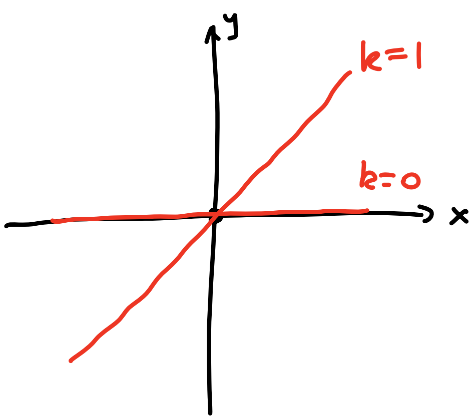
- Sum, difference, product, quotient (w/ denom $\neq 0$) of continuous functions are continuous
- Composition of continuous functions are continuous

For example, $f(x,y) = \dfrac{\ln(x^2+2y^2-1)}{x-2}$ is a continuous function in its domain $\{(x,y) : x \neq 2,\ x^2+2y^2-1 > 0\}$

$$\Rightarrow \lim_{(x,y)\to(4,1)} \frac{\ln(x^2+2y^2-1)}{x-2} = \frac{\ln(4^2+2\times 1^2-1)}{4-2} = \frac{\ln(17)}{2}$$

- Examples of "limit DNE" (NOT in exam)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$



Along $y = kx$

$$\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+(kx)^2} = \frac{x^2}{x^2+k^2x^2}$$

$$= \frac{1}{1+k^2} \longrightarrow \frac{1}{1+k^2} \quad (\text{as } x \to 0)$$

Result depends on k $\Rightarrow$ limit DNE.

4.3 Partial derivatives

Def The <u>partial derivative</u> of $f(x, y)$ with respect to x is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

The <u>partial derivative</u> of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

" $\frac{\partial f}{\partial x}$: taking derivative by viewing x as variable, y as constant "

Ex $f(x, y) = x^3 - x^2 y + 2\sin y$. Calculate :

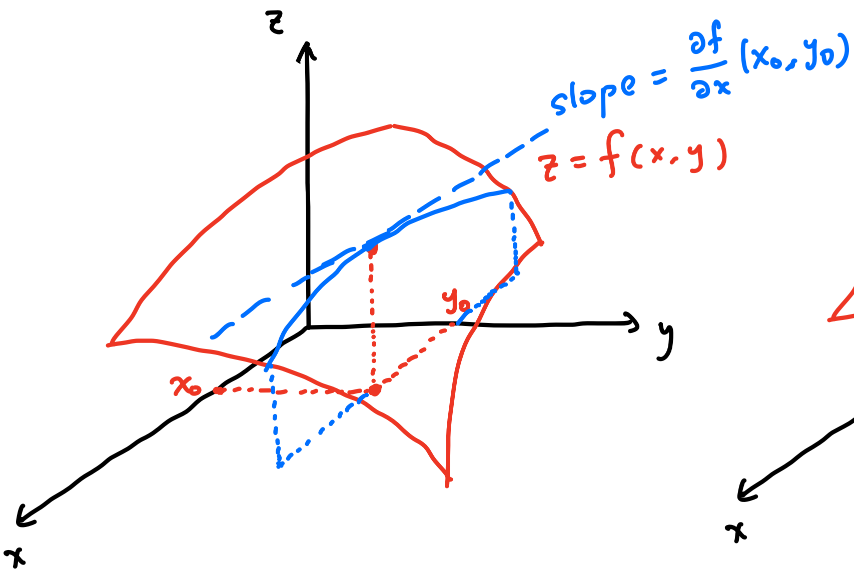
$$\frac{\partial f}{\partial x} = 3x^2 - y \cdot 2x \qquad \frac{\partial f}{\partial y} = -x^2 + 2\cos y$$

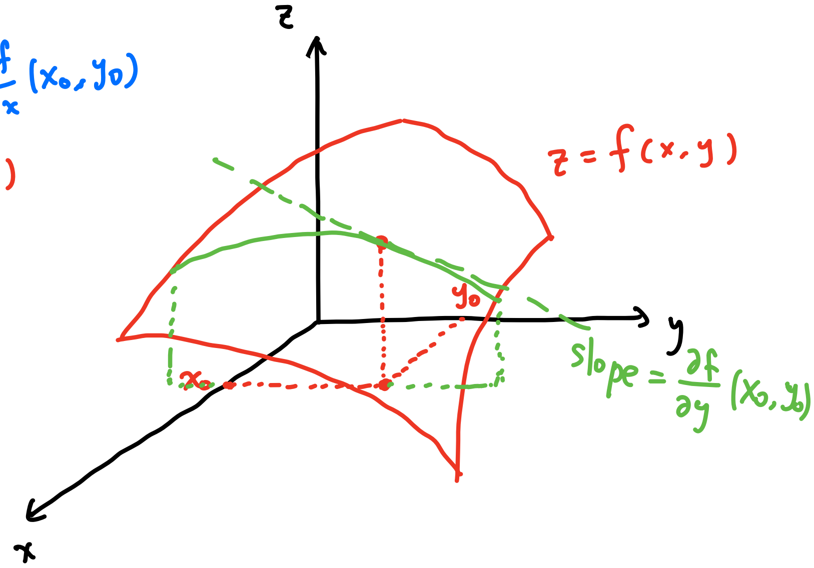
Ex $g(x, y) = e^{x^2+2y} \cos(xy^2)$

$$\frac{\partial g}{\partial x} = e^{x^2+2y} \cdot (2x) \cdot \cos(xy^2) + e^{x^2+2y} \cdot (-\sin(xy^2)) \cdot y^2$$

$$\frac{\partial g}{\partial y} = e^{x^2+2y} \cdot (2) \cdot \cos(xy^2) + e^{x^2+2y} \cdot (-\sin(xy^2)) \cdot 2xy$$

• Geometric meaning of partial derivatives





• Similarly define partial derivatives for $f(x,y,z)$

$\frac{\partial f}{\partial x}$: derivative w/ x as variable, y, z as constants

Ex $f(x,y,z) = \dfrac{x-y}{x^2+y^2+z^2}$

$$f_x = \frac{1\cdot(x^2+y^2+z^2) - (x-y)\cdot 2x}{(x^2+y^2+z^2)^2}$$

$$f_y = \frac{(-1)\cdot(x^2+y^2+z^2) - (x-y)\cdot 2y}{(x^2+y^2+z^2)^2}$$

$$f_z = \frac{0\cdot(x^2+y^2+z^2) - (x-y)\cdot 2z}{(x^2+y^2+z^2)^2}$$

• Higher order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

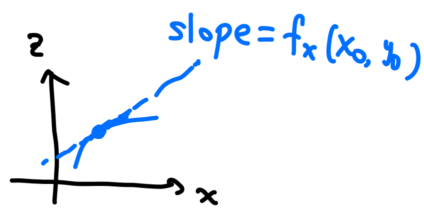
Thm If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$

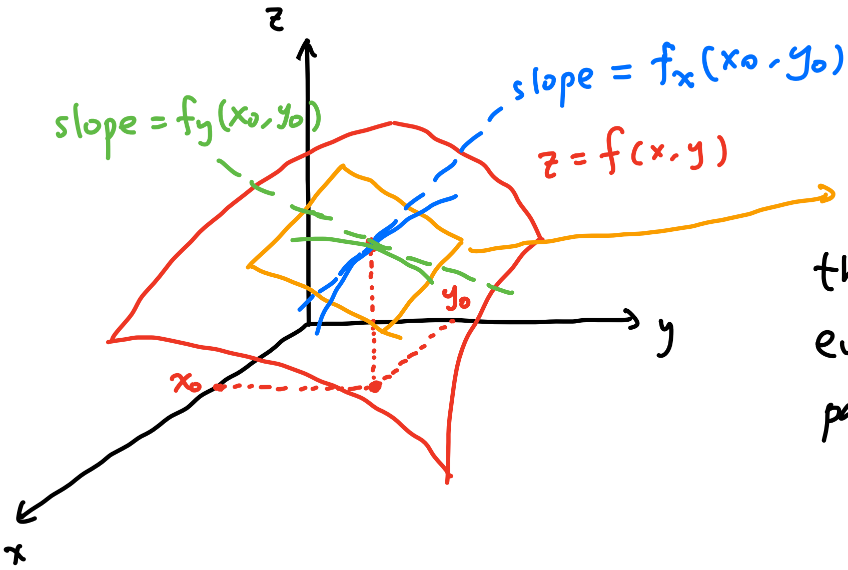
Ex $f(x,y) = x^2 - 2xy + y^3$ Calculate f_{xx}, f_{xy}, f_{yy}

$f_x = 2x - 2y$ $\qquad$ $f_y = -2x + 3y^2$

$f_{xx} = 2$ $\qquad$ $f_{xy} = -2$ $\qquad$ $f_{yy} = 6y$

4.4 Tangent planes and linear approximations





tangent plane at $(x_0, y_0, f(x_0, y_0))$

the plane that is tangent to every curve in $z = f(x,y)$ passing $(x_0, y_0, f(x_0,y_0))$

Tangent plane is parallel to $\langle 1, 0, f_x(x_0,y_0)\rangle$ $\langle 0, 1, f_y(x_0,y_0)\rangle$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(x_0,y_0) \\ 0 & 1 & f_y(x_0,y_0) \end{vmatrix} = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle$$

$\Rightarrow$ Equation of tangent plane :

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + \left(z - f(x_0, y_0)\right) = 0$$

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$