4.4 (continued)

Equation of tangent plane for $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Ex Find tangent plane of $z = \underbrace{x^2 + 2y^2}_{f(x,y)}$

① at $(\overset{x_0}{2}, \overset{y_0}{-1}, \overset{f(x_0,y_0)}{6})$

② at $(0, 0, 0)$

$f_x = 2x \qquad f_y = 4y$

At $(2, -1, 6)$, $f(2, -1) = 6$, $f_x(2, -1) = 4$, $f_y(2, -1) = -4$

$\Rightarrow \; z = 6 + 4(x - 2) - 4(y + 1)$

At $(0, 0, 0)$, $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$

$\Rightarrow \; z = 0$

- Linear approximation: use the tangent plane of $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ to approximate f near (x_0, y_0)

Def The linear approximation of $f(x, y)$ at (x_0, y_0) is

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

It approximates $f(x, y)$ when (x, y) is close to (x_0, y_0)

Ex Use linear approximation of $f(x,y) = x^3 y^2$ at $(1,2)$ to approximate $\underbrace{1.01^3 \times 1.98^2}_{f(1.01,\ 1.98)}$

$f_x = 3x^2y^2 \qquad f_y = 2x^3 y$

$f(1,2) = 4 \qquad f_x(1,2) = 12 \qquad f_y(1,2) = 4$

$\Rightarrow L(x,y) = 4 + 12(x-1) + 4(y-2)$

$$f(1.01, 1.98) \approx L(1.01, 1.98) = 4 + 12(1.01-1) + 4(1.98-2)$$

$$= 4 + 12 \times 0.01 + 4 \times (-0.02) = 4.04$$

error ≈ 0.001

$\sqrt{(x-x_0)^2 + (y-y_0)^2} = \sqrt{0.01^2 + 0.02^2} \approx 0.02$

exact value = 4.03919204

- Differentiability : justifying how good a linear approximation is.

Def $f(x,y)$ is <u>differentiable</u> at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

where $L(x,y)$ is the linear approximation of f at (x_0, y_0)

- Differentiability $\Rightarrow$ continuity
- If f, f_x, f_y are continuous, then f is differentiable.

- Differentials

$$f(x,y) - f(x_0,y_0) \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\Delta z \approx f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y$$

$$\Downarrow$$

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

"the total differential of $z = f(x,y)$ at (x_0, y_0)"

Ex Find total differential of the volume of a cylinder : $V = \pi r^2 h$ at $r=2$, $h=3$. Is its volume more sensitive to r or h ?

$V(r,h)$

$V_r = 2\pi r h$ $\quad\quad$ $V_h = \pi r^2$

$V_r(2,3) = 12\pi$ $\quad\quad$ $V_h(2,3) = 4\pi$

$\Rightarrow$ $dV = 12\pi\,dr + 4\pi\,dh$

$|12\pi| > |4\pi|$ $\Rightarrow$ more sensitive to r.

4.5 The chain rule

Recall : $\frac{d}{dt}\left(f(g(t))\right) = f'(g(t))\cdot g'(t) = \frac{df}{dg}\cdot\frac{dg}{dt}$

Thm $\frac{d}{dt}\left(f(x(t),y(t))\right) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$

evaluate at $(x(t), y(t))$

"proof"

$$f(x(t+\Delta t),\ y(t+\Delta t))$$

$$\approx f\left(x(t) + x'(t)\Delta t,\ y(t) + y'(t)\Delta t\right)$$

$$\approx f(x(t), y(t)) + f_x(x(t), y(t))\cdot x'(t)\Delta t + f_y(x(t), y(t))\cdot y'(t)\Delta t$$

(need differentiability to justify)

$\hookrightarrow \frac{d}{dt}\left(f(x(t), y(t))\right)$

Ex Values of f, f_x, f_y are given by

	f	f_x	f_y
$(0,0)$	1	-1	2
$(0,1)$	2	0	-2
$(1,1)$	3	-3	6

Calculate $\left.\frac{d}{dt}\right|_{t=0} f(t, t^2)$, $\left.\frac{d}{dt}\right|_{t=1} f(t, t^2)$

$x(t)=t$, $y(t)=t^2$

$$\frac{d}{dt} f(t, t^2) = f_x(t, t^2)\cdot 1 + f_y(t, t^2)\cdot 2t$$

$$\left.\frac{d}{dt}\right|_{t=0} f(t, t^2) = f_x(0,0)\cdot 1 + f_y(0,0)\cdot 2\cdot 0 = -1$$

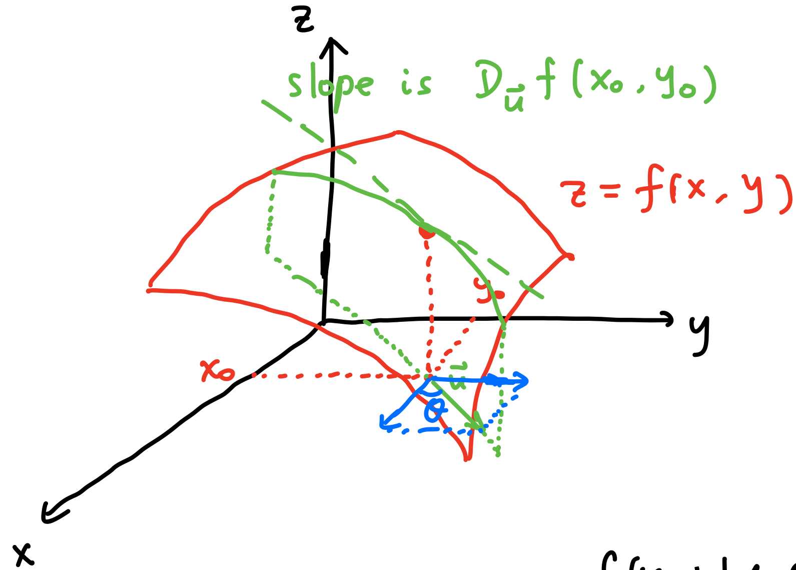
$$\left.\frac{d}{dt}\right|_{t=1} f(t, t^2) = f_x(1,1)\cdot 1 + f_y(1,1)\cdot 2\cdot 1 = -3 + 6\cdot 2 = 9$$

• Other version of chain rule

$$\frac{\partial}{\partial u}\Big(f\big(x(u,v),\, y(u,v)\big)\Big) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial}{\partial v}\Big(f\big(x(u,v),\, y(u,v)\big)\Big) = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

4.6 Directional derivatives and the gradient



Def Let $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. The directional derivative of f in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h\cos\theta,\ y_0 + h\sin\theta) - f(x_0, y_0)}{h}$$

Use linear approximation,

$$f(x_0 + h\cos\theta,\ y_0 + h\sin\theta) - f(x_0, y_0)$$

$$\approx f_x(x_0, y_0)\, h\cos\theta + f_y(x_0, y_0)\, h\sin\theta$$

$$= \big(f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta\big)\, h$$

$$\Rightarrow D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$ (justify by differentiability)

$$= \langle f_x(x_0, y_0),\, f_y(x_0, y_0) \rangle \cdot \vec{u}$$ $\vec{u} = \langle \cos\theta, \sin\theta \rangle$

Def The gradient of f is

$$\nabla f(x, y) = \langle f_x(x, y),\, f_y(x, y) \rangle$$

with this notation,

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

Ex Let $f(x,y) = 2x^2 - xy$, $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

① Find ∇f

② Find $D_{\vec{u}} f(1,-3)$

$f_x = 4x - y$ $\quad$ $f_y = -x$

$\nabla f = \langle 4x - y, -x \rangle$

$$D_{\vec{u}} f(1,-3) = \nabla f(1,-3) \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$= \langle 7, -1 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = 7 \cdot \frac{3}{5} - \frac{4}{5} = \frac{17}{5}$$