## 4.6 (continued)

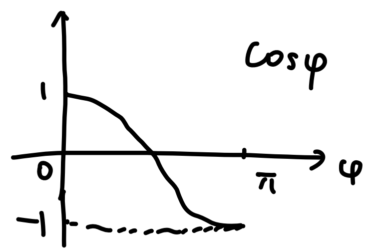
Recall: $\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

- Question: At $(x_0, y_0)$, in which direction does $f$ increase/decrease fastest? That is, find $\vec{u}$ such that $D_{\vec{u}} f(x_0, y_0)$ is maximized/minimized.

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u} = \|\nabla f(x_0, y_0)\| \cdot \|\vec{u}\| \cdot \cos\varphi$$

where $\varphi$ denotes the angle between $\nabla f(x_0, y_0)$ and $\vec{u}$.




$\Rightarrow$ $D_{\vec{u}} f(x_0, y_0)$ is largest when $\varphi = 0$,

that is, $\boxed{f \text{ increases fastest in } \nabla f(x_0, y_0) \text{ direction}}$

$D_{\vec{u}} f(x_0, y_0)$ is smallest when $\varphi = \pi$,

when $\nabla f(x_0, y_0) \neq \vec{0}$

that is, $\boxed{f \text{ decreases fastest in } -\nabla f(x_0, y_0) \text{ direction}}$

Ex Let $f(x,y) = 2x^2 - xy$. Find the direction in which $f$ decreases fastest, starting at $(1, -3)$. Describe the direction by unit vector.

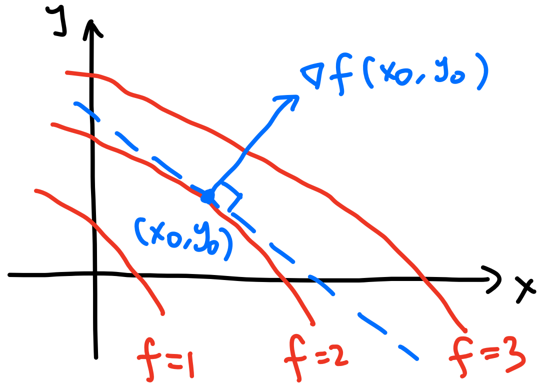
$\nabla f = \langle 4x - y, -x \rangle$ $\qquad$ $\nabla f(1, -3) = \langle 7, -1 \rangle$

$\Rightarrow$ $f$ decreases fastest in $\langle -7, 1 \rangle$ direction,

length $= \sqrt{(-7)^2 + 1^2}$
$= \sqrt{50}$

in unit vector, $\frac{1}{\sqrt{50}} \langle -7, 1 \rangle$

- $\nabla f(x_0, y_0)$ is perpendicular to the level curve of $f$ containing $(x_0, y_0)$




**Proof** Let $\langle x(t), y(t) \rangle$ be the vector eq of a level curve $f(x,y) = c$, w/ $\langle x(t_0), y(t_0) \rangle = \langle x_0, y_0 \rangle$

$$f(x(t), y(t)) = c$$

$$\Big\downarrow \frac{d}{dt}$$

$$f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t) = 0$$

$$\nabla f(x(t_0), y(t_0)) \cdot \langle x'(t_0), y'(t_0) \rangle = 0$$

## 4.7 Maxima / Minima problems

**Def** $f(x,y)$ has a local maximum at $(x_0, y_0)$ if

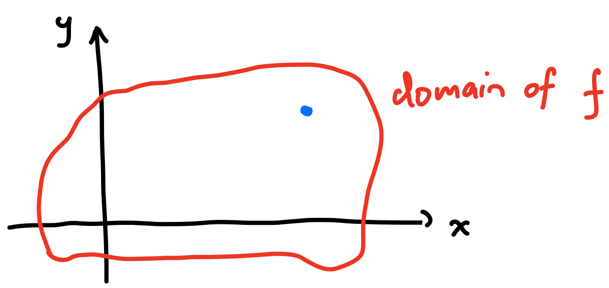

$$f(x,y) \le f(x_0, y_0)$$

for $(x,y)$ near $(x_0, y_0)$. The value $f(x_0, y_0)$ is a local maximal value.

If this inequality holds for any $(x,y)$ in the domain of $f$, then $f$ has a global (absolute) maximum at $(x_0, y_0)$.

Similarly define local / global minimum


- When using "local extremum" (that is, max or min), we usually require $f$ defined in a disk centered at $(x_0, y_0)$. For "global extremum", $(x_0, y_0)$ can be any pt in domain.



**Def** $(x_0, y_0)$ is a critical pt of $f(x,y)$ if $\nabla f(x_0, y_0) = \vec{0}$,

or $\nabla f(x_0, y_0)$ undefined

**Thm** If $(x_0, y_0)$ is a local extremum of $f$, then it is a critical pt of $f$

"Proof". Suppose $(x_0, y_0)$ is not a critical pt. Then $\nabla f(x_0, y_0) \neq \vec{0}$

$\Rightarrow$ Starting from $(x_0, y_0)$, $f$ increases in $\nabla f(x_0, y_0)$ direction,

decreases in $-\nabla f(x_0, y_0)$ direction

$\Rightarrow$ $(x_0, y_0)$ is not a local extremum.

**Ex** Find all critical pts of $f(x,y) = x^3 + xy - y$

$$\begin{cases} f_x = 3x^2 + y = 0 \\ f_y = x - 1 = 0 \quad \Rightarrow \quad x = 1 \end{cases}$$

$$y = -3x^2 = -3 \cdot 1^2 = -3$$

$\Rightarrow$ critical pt : $(1, -3)$

- Determine whether a critical pt is local max/min/neither:

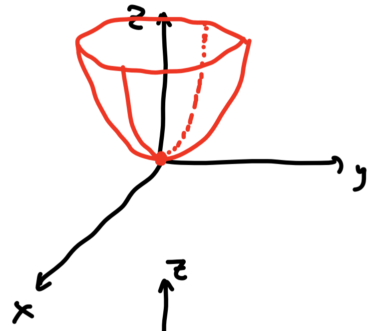
<u>Second derivative test</u>

$$D := \left( f_{xx} f_{yy} - f_{xy}^2 \right)\Big|_{(x,y)=(x_0,y_0)}$$

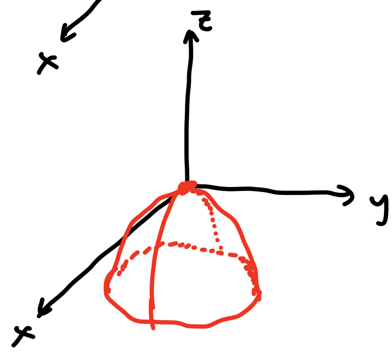
- If $D > 0$, $f_{xx}(x_0, y_0) > 0$, then $(x_0, y_0)$ is local min
- If $D > 0$, $f_{xx}(x_0, y_0) < 0$, then $(x_0, y_0)$ is local max
- If $D < 0$, then $(x_0, y_0)$ is neither.
- If $D = 0$, then this test is inconclusive.

Prototype (at $(x_0, y_0) = (0,0)$)

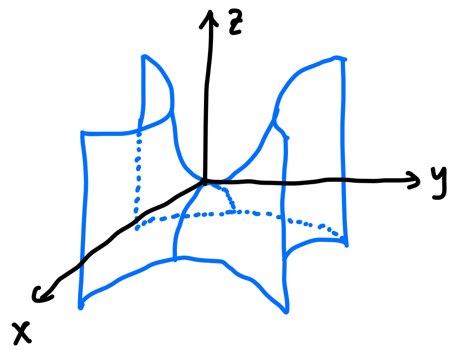
- $D > 0$, $f_{xx} > 0$: $f = x^2 + y^2$




- $D > 0$, $f_{xx} < 0$: $f = -x^2 - y^2$




- $D < 0$: $f = -x^2 + y^2$




<u>Ex</u> In the previous example, determine the type of the critical pt.

$$f(x,y) = x^3 + xy - y$$

$f_x = 3x^2 + y$

$f_y = x - 1$  critical pt: $(1, -3)$

$f_{xx} = 6x$  $f_{xy} = 1$  $f_{yy} = 0$

At $(1,-3)$: $f_{xx} = 6$  $f_{xy} = 1$  $f_{yy} = 0$

$$D = f_{xx} f_{yy} - f_{xy}^2 = 6 \cdot 0 - 1^2 = -1 < 0$$

$\Rightarrow$ neither.