4.7 (continued)

Recall:

Second derivative test

$$D := \left(f_{xx} f_{yy} - f_{xy}^2 \right)\Big|_{(x,y)=(x_0,y_0)}$$

- If $D > 0$, $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is local min
- If $D > 0$, $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is local max
- If $D < 0$, then (x_0, y_0) is neither.
- If $D = 0$, then this test is inconclusive.

Ex Find critical pts of $f(x, y) = x^4 - x^2 + y^2$ and determine type.

$$\begin{cases} f_x = 4x^3 - 2x = 0 & \Rightarrow 2x(2x^2 - 1) = 0 \quad x = 0, \pm \frac{1}{\sqrt{2}} \\ f_y = 2y = 0 & \Rightarrow y = 0 \end{cases}$$

$\Rightarrow$ critical pts : $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, 0\right)$, $\left(-\frac{1}{\sqrt{2}}, 0\right)$

$f_{xx} = 12x^2 - 2 \qquad f_{xy} = 0 \qquad f_{yy} = 2$

At $(0, 0)$: $f_{xx} = -2 \qquad f_{xy} = 0 \qquad f_{yy} = 2$

$D = (-2)\cdot 2 - 0^2 = -4 < 0 \Rightarrow$ neither

At $\left(\frac{1}{\sqrt{2}}, 0\right)$: $f_{xx} = 12\cdot\left(\frac{1}{\sqrt{2}}\right)^2 - 2 = 4 \qquad f_{xy} = 0 \qquad f_{yy} = 2$

$D = 4\cdot 2 - 0^2 = 8 > 0 \qquad f_{xx} = 4 > 0 \Rightarrow$ local min

At $\left(-\frac{1}{\sqrt{2}}, 0\right)$: $f_{xx} = 12\cdot\left(-\frac{1}{\sqrt{2}}\right)^2 - 2 = 4 \qquad f_{xy} = 0 \qquad f_{yy} = 2 \Rightarrow$ local min.

- How to find global max/min ?

Thm A continuous function defined on a closed and bounded domain achieves global max and min.

closed: domain contains its boundary

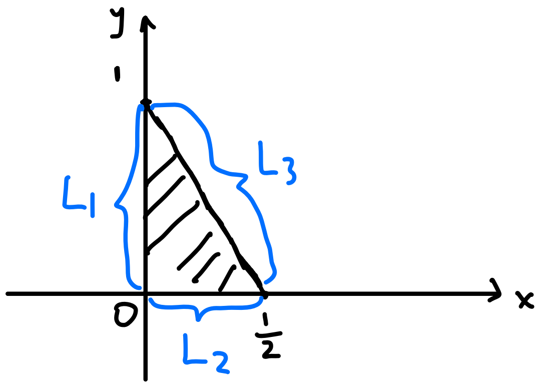
bounded: cannot extend to infinity

To find global max/min of f defined on closed and bounded D:

- Find critical pts of f in D
- Find max/min of f on the boundary of D (or find candidates of that)
- Largest among these f values $\Rightarrow$ global max

  Smallest among these f values $\Rightarrow$ global min

Ex Find global max/min of $f(x,y) = x^2 + 4xy + y$ on the closed region bounded by $x=0$, $y=0$, $y=-2x+1$



- Critical pts :

$$\begin{cases} f_x = 2x+4y = 0 \\ f_y = 4x+1 = 0 \end{cases} \qquad x = -\frac{1}{4}$$

$$4y = -2x = -2\cdot\left(-\frac{1}{4}\right) = \frac{1}{2}$$

$$y = \frac{1}{8} \quad \Rightarrow \left(-\frac{1}{4}, \frac{1}{8}\right) \text{ NOT in domain}$$

- L_1 : $x=0$, $0 \le y \le 1$

$f = y$ $\qquad y=1$: $f = \underline{1}$

$\qquad\qquad y=0$: $f = \underline{0}$

- L_2 : $y=0$, $0 \le x \le \frac{1}{2}$

$f = x^2$ $\qquad x = \frac{1}{2}$: $f = \underline{\frac{1}{4}}$

$\qquad\qquad x = 0$: $f = \underline{0}$

• L_3: $y=-2x+1,\ 0\le x\le \frac{1}{2}$

$$\begin{aligned} f &= x^2+4xy+y \\ &= x^2+4x(-2x+1)+(-2x+1) \\ &= x^2-8x^2+4x-2x+1 \\ &= -7x^2+2x+1 \end{aligned}$$

$\Big\downarrow \frac{d}{dx}$

$-14x+2=0 \Rightarrow x=\frac{1}{7}$

$x=0$: $f=1$

$x=\frac{1}{2}$: $f=-7\cdot\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}+1=\frac{1}{4}$

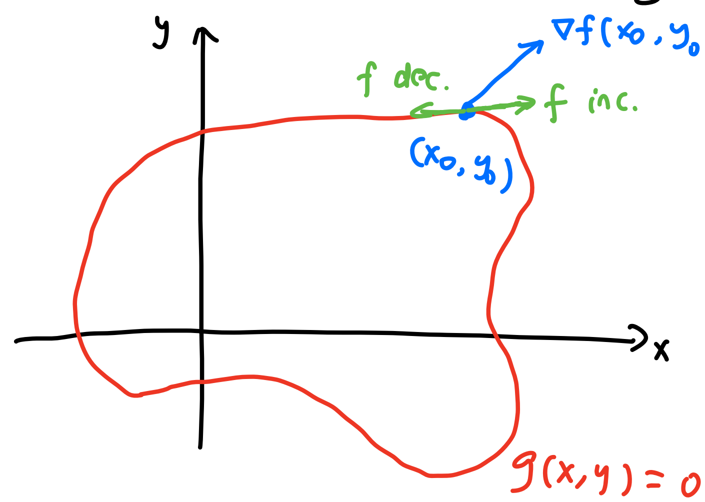
$x=\frac{1}{7}$: $f=-7\cdot\left(\frac{1}{7}\right)^2+2\cdot\frac{1}{7}+1=\frac{8}{7}$

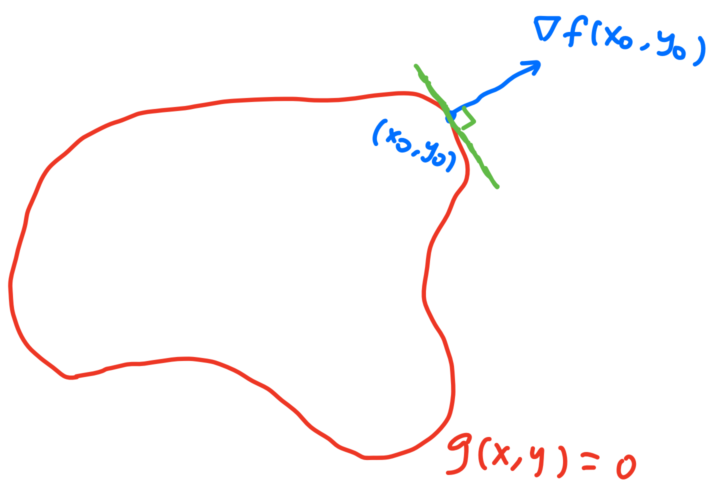
$\Rightarrow$ global max: $\frac{8}{7}$, achieved at $\left(\frac{1}{7},\frac{5}{7}\right)$

global min: 0, achieved at $(0,0)$

4.8 Lagrange multipliers

Constraint optimization problem: find max/min of $f(x,y)$, subject to the constraint $g(x,y)=0$





Thm With constraint $g(x,y)=0$, if f achieves max or min at (x_0,y_0) and $\nabla g(x_0,y_0)\ne\vec{0}$, then there exists λ such that

$$\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

Ex Find max/min of $f(x,y)=6x-4y$ subject to $x^2+y^2=-2x$

$g(x,y)=x^2+y^2+2x=0$

$$\nabla f = \langle 6, -4 \rangle \qquad \nabla g = \langle 2x+2, 2y \rangle$$

$$\langle 6, -4 \rangle = \lambda \langle 2x+2, 2y \rangle$$

$$\begin{cases} 6 = \lambda(2x+2) \\ -4 = \lambda \cdot 2y \\ x^2+y^2+2x = 0 \end{cases}$$

In the exam, you can stop here.

$$\frac{6}{\lambda} = 2x+2 \qquad y = -\frac{4}{2\lambda} = -\frac{2}{\lambda}$$

$$\frac{3}{\lambda} = x+1$$

$$x = \frac{3}{\lambda} - 1$$

$$\left(\frac{3}{\lambda}-1\right)^2 + \left(-\frac{2}{\lambda}\right)^2 + 2\left(\frac{3}{\lambda}-1\right) = 0$$

$$\frac{9}{\lambda^2} - \frac{6}{\lambda} + 1 + \frac{4}{\lambda^2} + \frac{6}{\lambda} - 2 = 0$$

$$\frac{13}{\lambda^2} - 1 = 0$$

$$\begin{cases} \lambda = \sqrt{13} \\ x = \frac{3}{\sqrt{13}} - 1 \\ y = -\frac{2}{\sqrt{13}} \end{cases} \quad \text{or} \quad \begin{cases} \lambda = -\sqrt{13} \\ x = -\frac{3}{\sqrt{13}} - 1 \\ y = \frac{2}{\sqrt{13}} \end{cases}$$

$$f = 6\left(\frac{3}{\sqrt{13}} - 1\right) - 4\cdot\left(-\frac{2}{\sqrt{13}}\right) = \frac{26}{\sqrt{13}} - 6$$

global max

$$f = 6\left(-\frac{3}{\sqrt{13}} - 1\right) - 4\cdot\frac{2}{\sqrt{13}} = -\frac{26}{\sqrt{13}} - 6$$

global min.