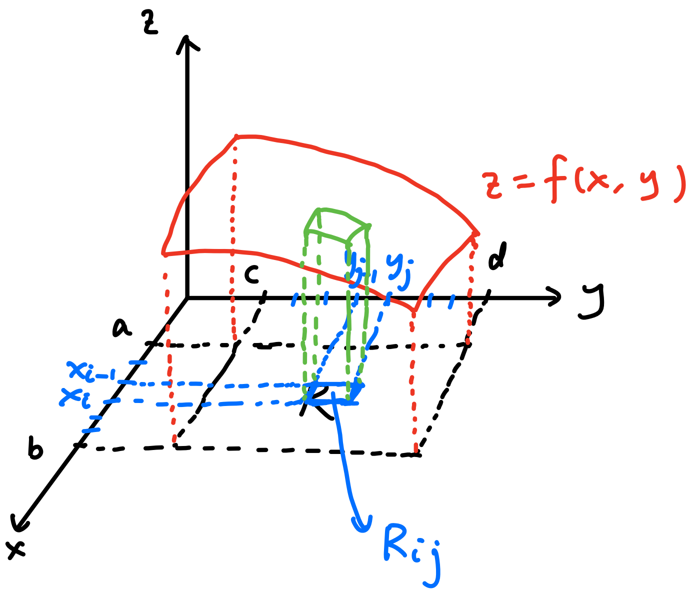
# 5.1 Double integrals over rectangular regions




Want the volume below $z = f(x,y)$ on $R = [a, b] \times [c, d]$

Partition: $a = x_0 < x_1 < \cdots < x_m = b$

$c = y_0 < y_1 < \cdots < y_n = d$

Spacing $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$

For $R_{ij}$: $x_{i-1} \le x \le x_i$, $y_{j-1} \le y \le y_j$, take $(x_{ij}^*, y_{ij}^*)$ in $R_{ij}$, approximate volume above $R_{ij}$ by

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A \qquad \Delta A = \Delta x \cdot \Delta y$$

Then sum over $i, j$ and take limit

**Def** The <u>double integral</u> of $f(x,y)$ over $R = [a,b] \times [c,d]$ is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \underbrace{\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}_{\text{"double Riemann sum"}}$$

also written as

$$\iint_R f(x,y)\,dx\,dy$$

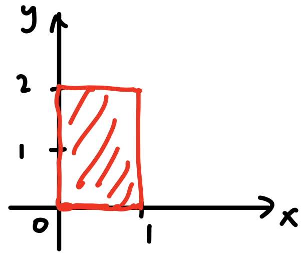
or $$\iint_R f(x,y)\,dy\,dx$$

• Calculate double integral by iterated integral (Fubini Thm)

Thm If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex $R = [0,1] \times [0,2]$




$\iint_R x^2 y\,dA$

$$= \int_0^1 \int_0^2 x^2 y\,dy\,dx = \int_0^1 \frac{1}{2}x^2y^2 \Big|_{y=0}^{2}\,dx$$

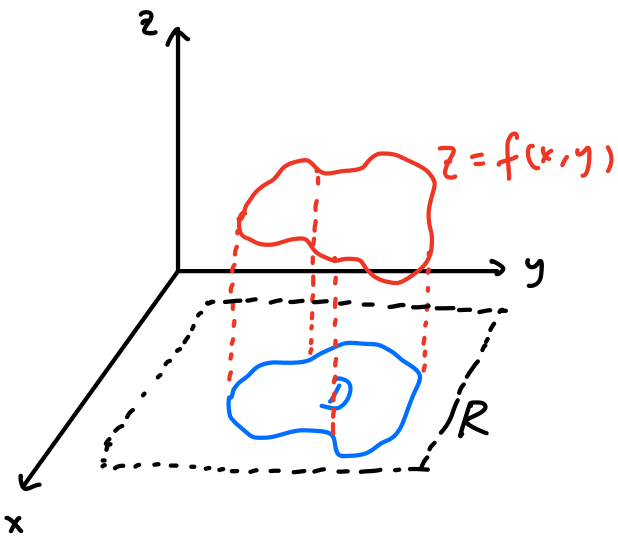
$$= \int_0^1 \left(\frac{1}{2}x^2\cdot 2^2 - \frac{1}{2}x^2\cdot 0^2\right)dx = \int_0^1 2x^2\,dx = 2\cdot\frac{1}{3}x^3\Big|_{x=0}^{1} = \frac{2}{3}$$

Another way:

$$\iint_R x^2y\,dA = \int_0^2\int_0^1 x^2y\,dx\,dy = \int_0^2 \frac{1}{3}x^3y\Big|_{x=0}^{1}\,dy$$

$$= \int_0^2\left(\frac{1}{3}\cdot 1^3\cdot y - \frac{1}{3}\cdot 0^3\cdot y\right)dy = \int_0^2 \frac{1}{3}y\,dy = \frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{y=0}^{2} = \frac{2}{3}$$

## 5.2 Double integrals over general regions




Let $D$ be a general region,

Suppose $D$ is inside a rectangle $R$

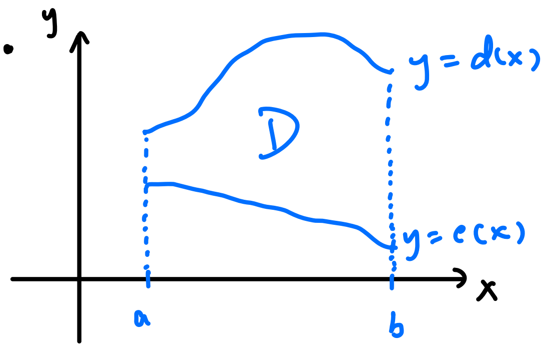
Then let

$$g(x,y) = \begin{cases} f(x,y), & (x,y) \text{ in } D \\ 0, & \text{elsewhere} \end{cases}$$

and define

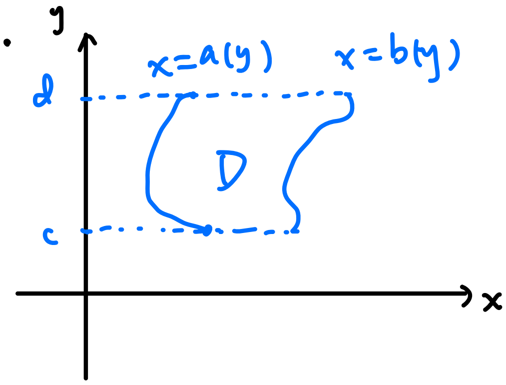
$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA$$

- Convert into iterated integral (2 cases)

- 


$D = \{(x,y) : a \le x \le b,\ c(x) \le y \le d(x)\}$

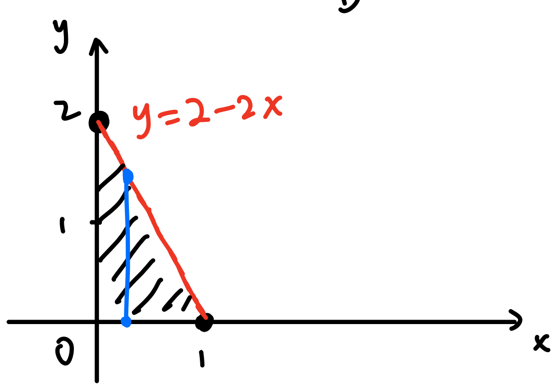
$$\iint_D f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$$

- 


$D = \{(x,y) : c \le y \le d,\ a(y) \le x \le b(y)\}$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\,dx\,dy$$

__Ex__ Calculate $\iint_D xy\,dA$, where $D$ is bounded by $x=0$, $y=0$, $y=2-2x$




$$\int_0^1 \int_0^{2-2x} xy \; dy\,dx$$

$$= \int_0^1 \frac{1}{2} x y^2 \Big|_{y=0}^{2-2x} dx$$

$$= \int_0^1 \frac{1}{2} x (2-2x)^2\,dx$$

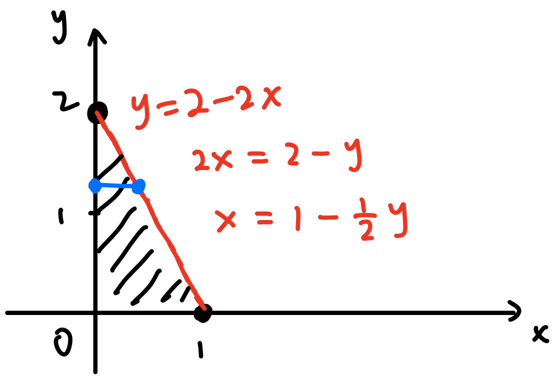
$$= \int_0^1 \frac{1}{2} x (4 - 8x + 4x^2)\,dx$$

$$= \int_0^1 (2x - 4x^2 + 2x^3)\,dx$$

$$= \left(2\cdot\frac{1}{2}x^2 - 4\cdot\frac{1}{3}x^3 + 2\cdot\frac{1}{4}x^4\right)\Big|_{x=0}^{1}$$

$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$

Another way :




$$\int_0^2 \int_0^{1-\frac{1}{2}y} xy \, dx\, dy$$

$$= \int_0^2 \frac{1}{2}x^2 y \Big|_{x=0}^{1-\frac{1}{2}y} dy$$

$$= \int_0^2 \frac{1}{2}\left(1-\frac{1}{2}y\right)^2 y \, dy$$

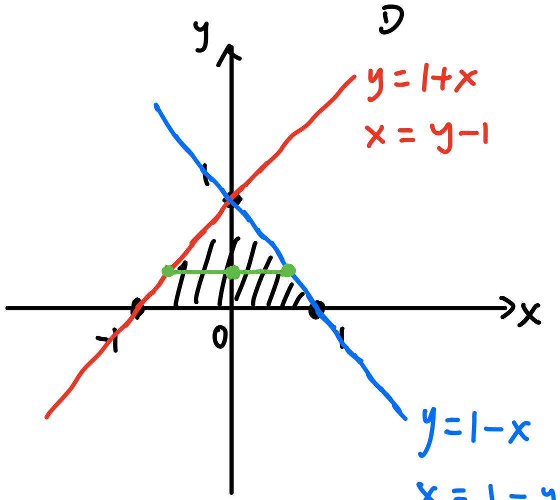
$$= \frac{1}{2}\int_0^2 \left(1-y+\frac{1}{4}y^2\right) y \, dy$$

$$= \frac{1}{2}\int_0^2 \left(y - y^2 + \frac{1}{4}y^3\right) dy$$

$$= \frac{1}{2}\left(\frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4}\cdot\frac{1}{4}y^4\right)\Big|_{y=0}^{2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{3}\cdot 8 + \frac{1}{16}\cdot 16\right) = \frac{1}{6}$$

Ex Calculate $\iint_D (y+1)\,dA$, where $D$ is bounded by $y=0$, $y=1+x$, $y=1-x$




$$\int_0^1 \int_{y-1}^{1-y} (y+1)\,dx\,dy$$

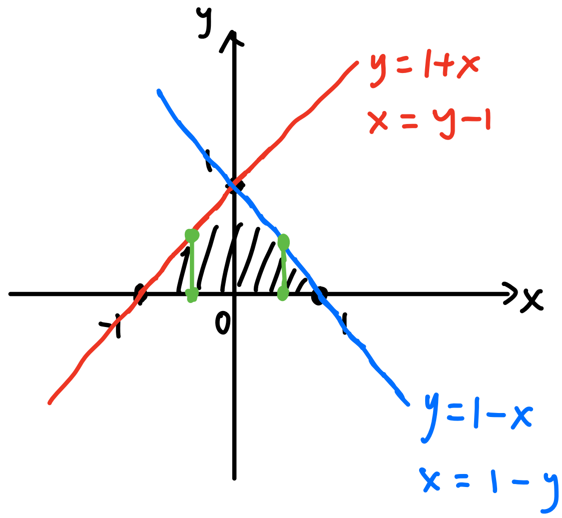
$$= \int_0^1 (y+1)\,x \Big|_{x=y-1}^{1-y} dy$$

$$= \int_0^1 (y+1)\big((1-y)-(y-1)\big)\,dy$$

$$= \int_0^1 (y+1)(2-2y)\,dy$$

$$= \int_0^1 (2-2y^2)\,dy = \left(2y - 2\cdot\frac{1}{3}y^3\right)\Big|_{y=0}^{1} = \frac{4}{3}$$

Another way:




$$\int_{-1}^{1} \int \qquad dy\,dx$$