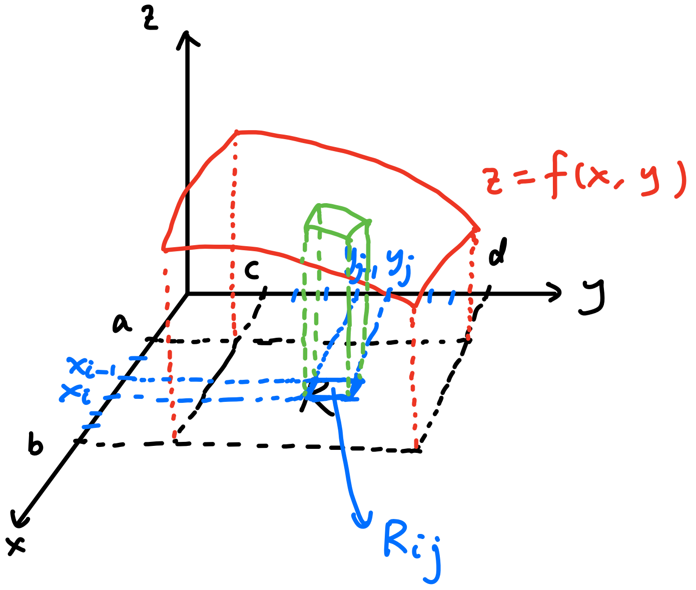
5.1 Double integrals over rectangular regions



Want the volume below $z=f(x,y)$ on $R=[a,b]\times[c,d]$

Partition: $a=x_0<x_1<\cdots<x_m=b$

$c=y_0<y_1<\cdots<y_n=d$

Spacing $\Delta x=\frac{b-a}{m}$, $\Delta y=\frac{d-c}{n}$

For R_{ij}: $x_{i-1}\le x\le x_i$, $y_{j-1}\le y\le y_j$, take (x_{ij}^*, y_{ij}^*) in R_{ij},

approximate volume above R_{ij} by

$$f(x_{ij}^*, y_{ij}^*)\,\Delta A \qquad \Delta A=\Delta x\cdot\Delta y$$

Then sum over i,j and take limit

<u>Def</u> The <u>double integral</u> of $f(x,y)$ over $R=[a,b]\times[c,d]$ is

$$\iint_R f(x,y)\,dA=\lim_{m,n\to\infty}\underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}_{\text{"double Riemann sum"}}$$

also written as

$$\iint_R f(x,y)\,dx\,dy$$

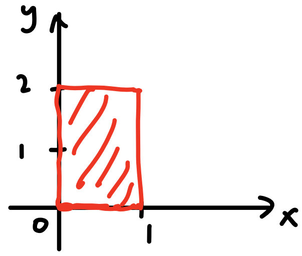
or $$\iint_R f(x,y)\,dy\,dx$$

• Calculate double integral by iterated integral (Fubini Thm)

Thm If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex $R=[0,1]\times[0,2]$



$$\iint_R x^2y\,dA$$

$$= \int_0^1\int_0^2 x^2y\,dy\,dx = \int_0^1 \frac{1}{2}x^2y^2\Big|_{y=0}^{2}\,dx$$

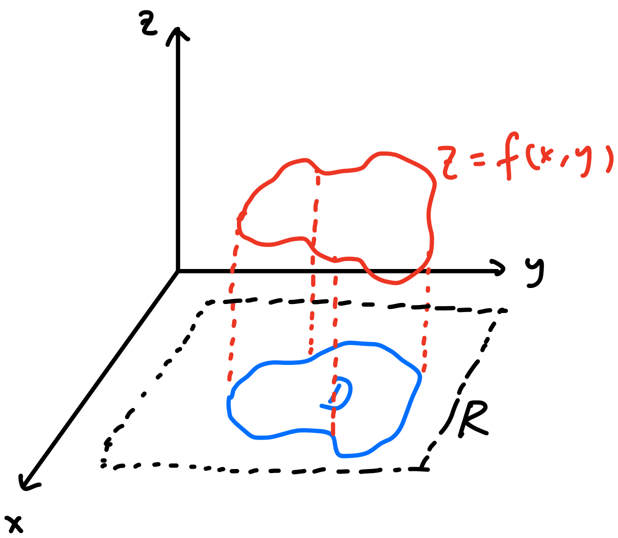
$$= \int_0^1\left(\frac{1}{2}x^2\cdot 2^2 - \frac{1}{2}x^2\cdot 0^2\right)dx = \int_0^1 2x^2\,dx = 2\cdot\frac{1}{3}x^3\Big|_{x=0}^{1} = \frac{2}{3}$$

Another way:

$$\iint_R x^2y\,dA = \int_0^2\int_0^1 x^2y\,dx\,dy = \int_0^2 \frac{1}{3}x^3y\Big|_{x=0}^{1}\,dy$$

$$= \int_0^2\left(\frac{1}{3}\cdot 1^3\cdot y - \frac{1}{3}\cdot 0^3\cdot y\right)dy = \int_0^2 \frac{1}{3}y\,dy = \frac{1}{3}\cdot\frac{1}{2}y^2\Big|_{y=0}^{2} = \frac{2}{3}$$

5.2 Double integrals over general regions



Let D be a general region.

Suppose D is inside a rectangle R

Then let

$$g(x,y) = \begin{cases} f(x,y), & (x,y) \text{ in } D \\ 0, & \text{elsewhere} \end{cases}$$

and define

$$\iint_D f(x,y)\,dA = \iint_R g(x,y)\,dA$$

· Convert into iterated integral (2 cases)

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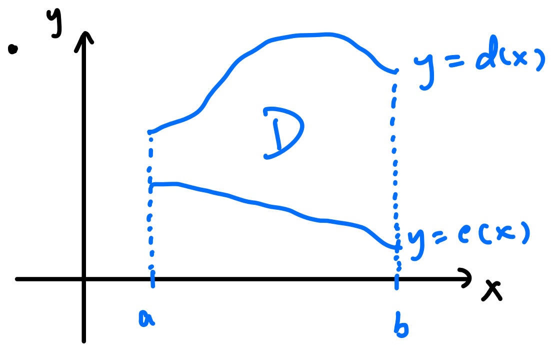

$D = \{(x, y) : a \le x \le b,\ c(x) \le y \le d(x)\}$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\, dx$$

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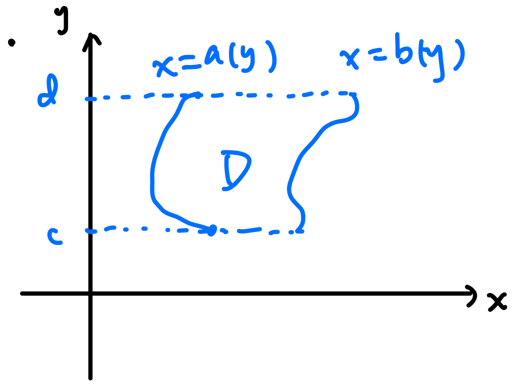
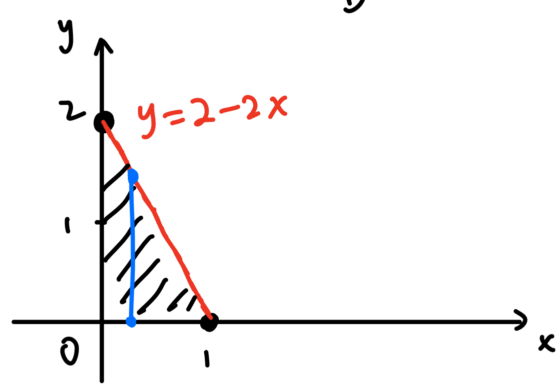


$D = \{(x, y) : c \le y \le d,\ a(y) \le x \le b(y)\}$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

<u>Ex</u> Calculate $\iint_D xy\, dA$, where D is bounded by $x = 0$, $y = 0$, $y = 2 - 2x$



$$\int_0^1 \int_0^{2-2x} xy \quad dy\, dx$$

$$= \int_0^1 \frac{1}{2} x y^2 \Big|_{y=0}^{2-2x} dx$$

$$= \int_0^1 \frac{1}{2} x (2-2x)^2\, dx$$

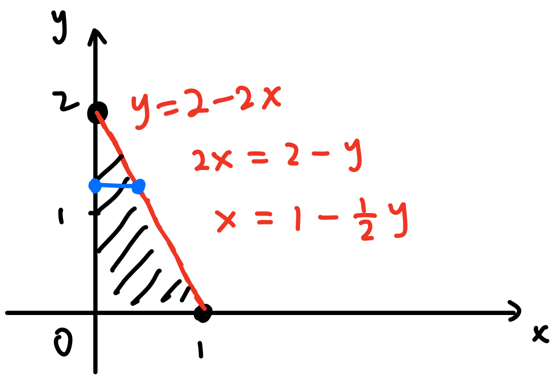
$$= \int_0^1 \frac{1}{2} x (4 - 8x + 4x^2)\, dx$$

$$= \int_0^1 (2x - 4x^2 + 2x^3)\, dx$$

$$= \left(2 \cdot \frac{1}{2} x^2 - 4 \cdot \frac{1}{3} x^3 + 2 \cdot \frac{1}{4} x^4\right) \Big|_{x=0}^{1}$$

$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$

Another way :



$$\int_0^2 \int_0^{1-\frac{1}{2}y} xy \, dx\, dy$$

$$= \int_0^2 \frac{1}{2}x^2 y \Big|_{x=0}^{1-\frac{1}{2}y} dy$$

$$= \int_0^2 \frac{1}{2}\left(1-\frac{1}{2}y\right)^2 y \, dy$$

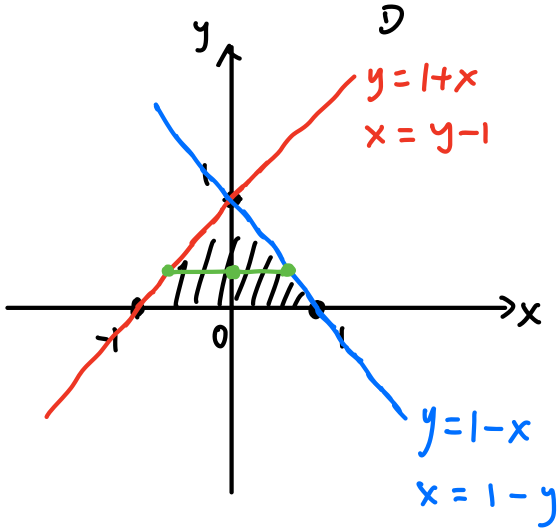
$$= \frac{1}{2}\int_0^2 \left(1-y+\frac{1}{4}y^2\right) y \, dy$$

$$= \frac{1}{2}\int_0^2 \left(y - y^2 + \frac{1}{4}y^3\right) dy$$

$$= \frac{1}{2}\left(\frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4}\cdot\frac{1}{4}y^4\right)\Big|_{y=0}^{2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot 4 - \frac{1}{3}\cdot 8 + \frac{1}{16}\cdot 16\right) = \frac{1}{6}$$

__Ex__ Calculate $\iint_D (y+1)\,dA$, where D is bounded by $y=0$, $y=1+x$, $y=1-x$



$$\int_0^1 \int_{y-1}^{1-y} (y+1)\,dx\,dy$$

$$= \int_0^1 (y+1)\,x \Big|_{x=y-1}^{1-y} dy$$

$$= \int_0^1 (y+1)\big((1-y)-(y-1)\big)\,dy$$

$$= \int_0^1 (y+1)(2-2y)\,dy$$

$$= \int_0^1 (2-2y^2)\,dy = \left(2y - 2\cdot\frac{1}{3}y^3\right)\Big|_{y=0}^{1} = \frac{4}{3}$$

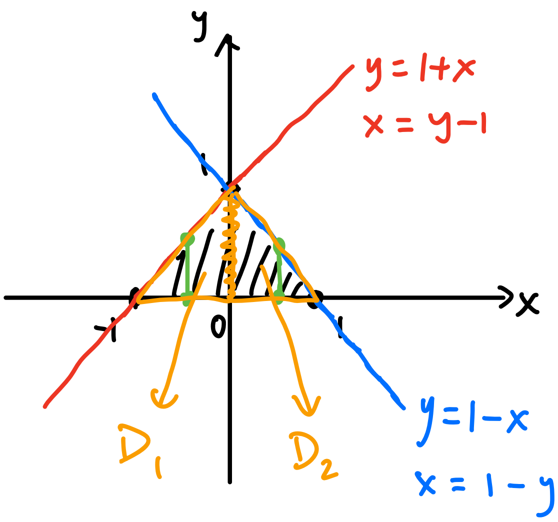
__Properties of double integrals__

- Linear property : $\iint_D \left(c_1 f(x,y) + c_2 g(x,y)\right) dA$

$$= c_1 \iint_D f(x,y)\,dA + c_2 \iint_D g(x,y)\,dA$$

- If $D = D_1 \cup D_2$ where D_1, D_2 do not overlap except for boundary, then $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

Another way for previous example:



$$\iint_D (y+1)\,dA = \iint_{D_1} (y+1)\,dA + \iint_{D_2} (y+1)\,dA$$

$$\iint_{D_1} (y+1)\,dA = \int_{-1}^{0}\int_0^{1+x} (y+1)\,dy\,dx$$

$$= \int_{-1}^{0} \left(\tfrac{1}{2}y^2 + y\right)\Big|_{y=0}^{1+x} dx$$

$$= \int_{-1}^{0} \left(\tfrac{1}{2}(1+x)^2 + (1+x)\right) dx$$

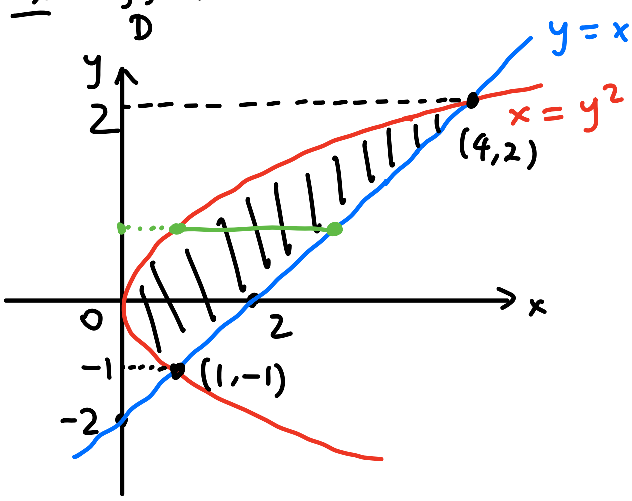
$$= \int_{-1}^{0} \left(\tfrac{1}{2} + x + \tfrac{1}{2}x^2 + 1 + x\right) dx = \int_{-1}^{0} \left(\tfrac{3}{2} + 2x + \tfrac{1}{2}x^2\right) dx$$

$$= \left(\tfrac{3}{2}x + x^2 + \tfrac{1}{2}\cdot\tfrac{1}{3}x^3\right)\Big|_{x=-1}^{0} = -\left(-\tfrac{3}{2} + 1 - \tfrac{1}{6}\right) = \tfrac{2}{3}.$$

$$\iint_{D_2} (y+1)\,dA = \int_0^1\int_0^{1-x} (y+1)\,dy\,dx = \cdots = \tfrac{2}{3}$$

$$\Rightarrow \iint_D (y+1)\,dA = \tfrac{2}{3} + \tfrac{2}{3} = \tfrac{4}{3}$$

Ex $\iint_D x\,dA$, where D is bounded by $x = y^2$ and $y = x-2$



y as outer

$$\int_{-1}^{2}\int_{y^2}^{y+2} x \,dx\,dy$$

$$= \int_{-1}^{2} \tfrac{1}{2}x^2 \Big|_{x=y^2}^{y+2} dy$$

$$= \tfrac{1}{2}\int_{-1}^{2} \left((y+2)^2 - y^4\right) dy$$

Calculate intersection pts:

$$\begin{cases} x = y^2 \\ y = x - 2 \end{cases}$$

$y = y^2 - 2 \qquad y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

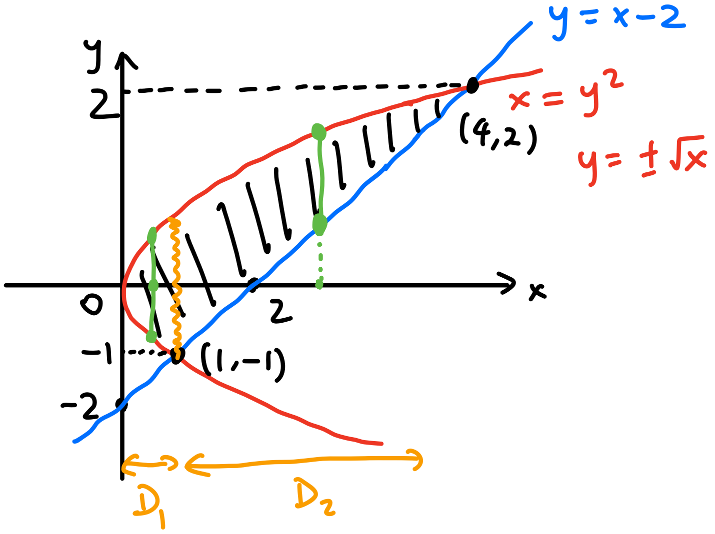
$y_1 = 2, \qquad y_2 = -1$

$x_1 = 4, \qquad x_2 = 1$

$$= \frac{1}{2}\left(\frac{1}{3}(y+2)^3 - \frac{1}{5}y^5 \right)\Bigg|_{y=-1}^{2}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}\cdot 4^3 - \frac{1}{5}\cdot 2^5 \right) - \left(\frac{1}{3} + \frac{1}{5} \right) \right] = \frac{36}{5}$$

Another way: x as outer



$$\iint_{D_1} x\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} x \, dy\, dx$$

$$= \cdots = \frac{4}{5}$$

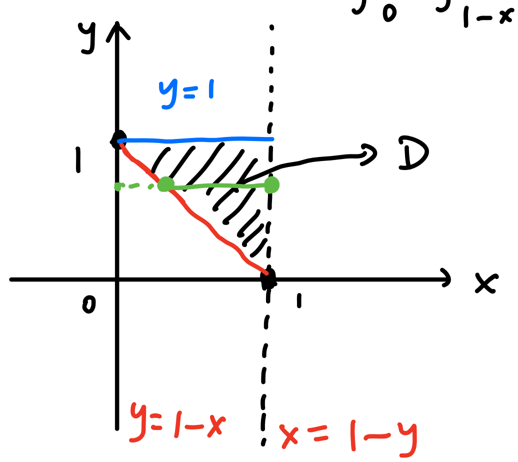
$$\iint_{D_2} x\, dA = \int_1^4 \int_{x-2}^{\sqrt{x}} x \, dy\, dx$$

$$= \cdots = \frac{32}{5}$$

- Change the order of iterated integral

$$\iint \cdots dy\, dx \longrightarrow \iint_D \cdots dA \longrightarrow \iint \cdots dx\, dy$$

<u>Ex</u> Calculate $\int_0^1 \int_{1-x}^1 e^{y^2}\, dy\, dx$



$$= \iint_D e^{y^2}\, dA = \int_0^1 \int_{1-y}^1 e^{y^2}\, dx\, dy$$

$$= \int_0^1 e^{y^2}\, x \Big|_{x=1-y}^{1}\, dy$$

$$= \int_0^1 e^{y^2}\left(1 - (1-y)\right) dy$$

$$= \int_0^1 e^{y^2}\, y\, dy = \frac{1}{2} e^{y^2} \Big|_{y=0}^{1} = \frac{1}{2}(e-1)$$

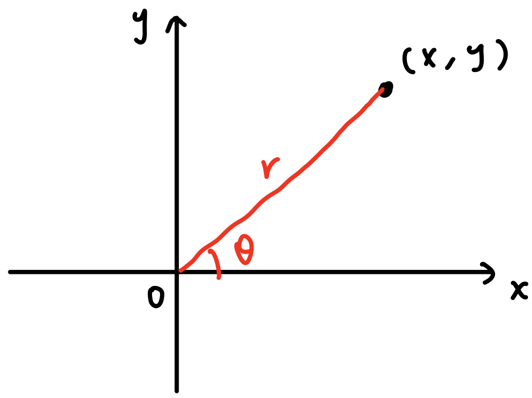
$$\int e^{y^2} y\, dy = \frac{1}{2}\int e^u\, du$$

$u = y^2$ $\qquad = \frac{1}{2} e^u + C$

$du = 2y\, dy$ $\qquad = \frac{1}{2} e^{y^2} + C$

5.3 Double integrals in polar coordinates

Recall polar coordinates:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

<u>Thm</u> If a region D (in (x, y)) can be described by "(r, θ) in R" for some region R, then

$$\iint_D f(x,y)\, dx\, dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

pay attention! "Jacobian"

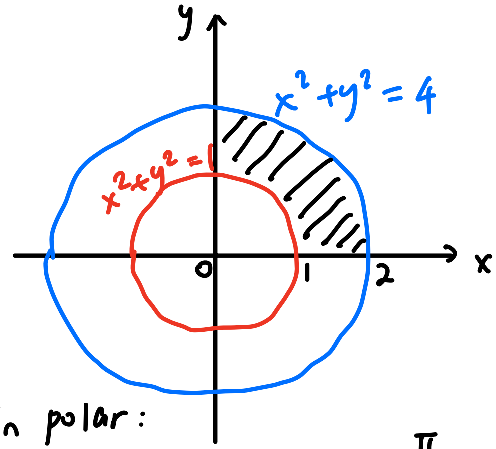
Compare w/ u-sub:

$$\int_a^b f(x)\, dx = \int_{..}^{..} f(g(u))\, g'(u)\, du$$

$x = g(u)$

$dx = g'(u)\, du$

<u>Ex</u> Let D be the region between $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ in the first quadrant. Calculate $\iint_D y\, dA$.



in polar:
$1 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2}$

$$= \int_1^2 \int_0^{\frac{\pi}{2}} r\sin\theta \cdot r \, d\theta \, dr$$

$$= \int_1^2 r^2 \cdot (-\cos\theta) \Big|_{\theta=0}^{\frac{\pi}{2}} dr$$

$$= \int_1^2 r^2 \cdot 1 \, dr = \frac{1}{3} r^3 \Big|_{r=1}^{2} = \frac{7}{3} .$$