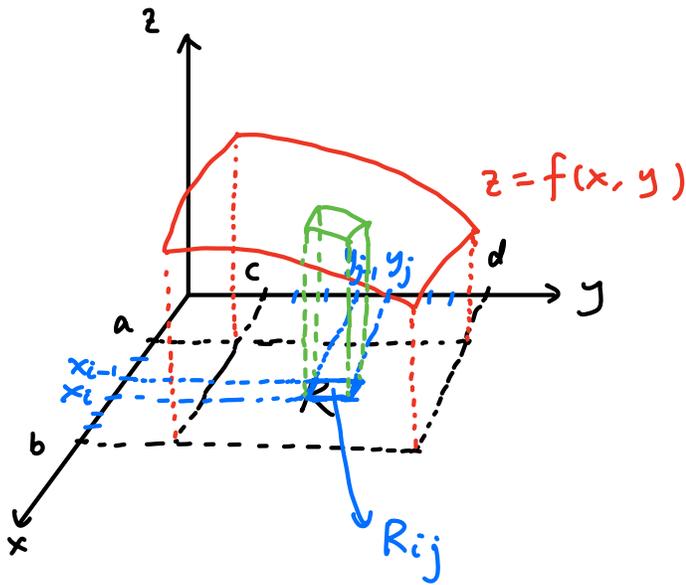


5.1 Double integrals over rectangular regions



Want the volume below $z = f(x, y)$
on $R = [a, b] \times [c, d]$

Partition: $a = x_0 < x_1 < \dots < x_m = b$

$c = y_0 < y_1 < \dots < y_n = d$

Spacing $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$

For R_{ij} : $x_{i-1} \leq x \leq x_i$, $y_{j-1} \leq y \leq y_j$, take (x_{ij}^*, y_{ij}^*) in R_{ij} ,
approximate volume above R_{ij} by

$$f(x_{ij}^*, y_{ij}^*) \Delta A$$

$$\Delta A = \Delta x \cdot \Delta y$$

Then sum over i, j and take limit

Def The double integral of $f(x, y)$ over $R = [a, b] \times [c, d]$ is

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \underbrace{\sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A}_{\text{"double Riemann sum"}}$$

also written as

$$\iint_R f(x, y) dx dy$$

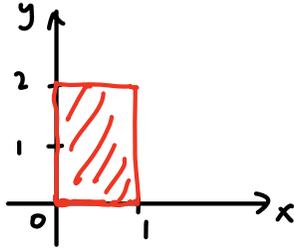
$$\text{or } \iint_R f(x, y) dy dx$$

• Calculate double integral by iterated integral (Fubini Thm)

Thm If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Ex $R = [0, 1] \times [0, 2]$



$$\iint_R x^2 y dA$$

$$= \int_0^1 \int_0^2 x^2 y dy dx = \int_0^1 \left. \frac{1}{2} x^2 y^2 \right|_{y=0}^2 dx$$

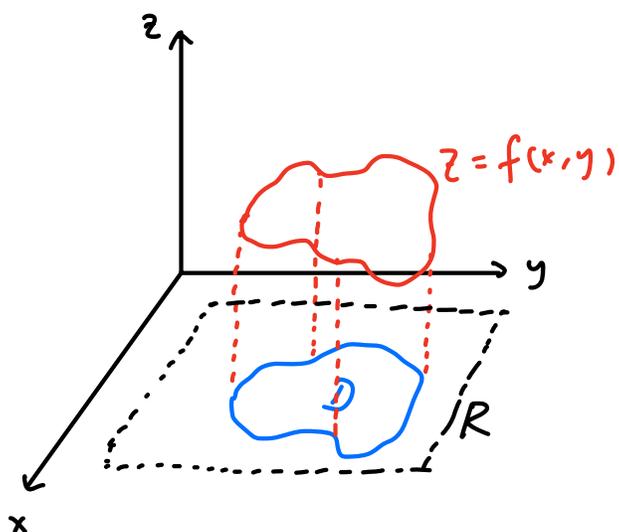
$$= \int_0^1 \left(\frac{1}{2} x^2 \cdot 2^2 - \frac{1}{2} x^2 \cdot 0^2 \right) dx = \int_0^1 2x^2 dx = 2 \cdot \frac{1}{3} x^3 \Big|_{x=0}^1 = \frac{2}{3}$$

Another way:

$$\iint_R x^2 y dA = \int_0^2 \int_0^1 x^2 y dx dy = \int_0^2 \left. \frac{1}{3} x^3 y \right|_{x=0}^1 dy$$

$$= \int_0^2 \left(\frac{1}{3} \cdot 1^3 \cdot y - \frac{1}{3} \cdot 0^3 \cdot y \right) dy = \int_0^2 \frac{1}{3} y dy = \frac{1}{3} \cdot \frac{1}{2} y^2 \Big|_{y=0}^2 = \frac{2}{3}$$

5.2 Double integrals over general regions



Let D be a general region,

Suppose D is inside a rectangle R

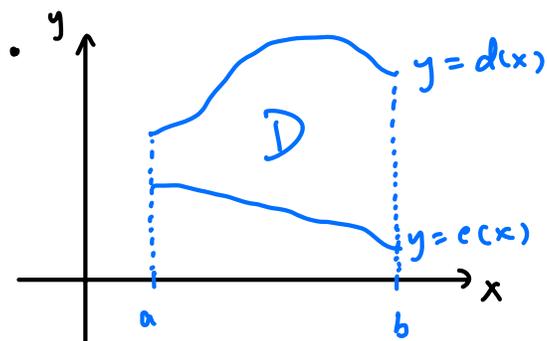
Then let

$$g(x, y) = \begin{cases} f(x, y), & (x, y) \text{ in } D \\ 0, & \text{elsewhere} \end{cases}$$

and define

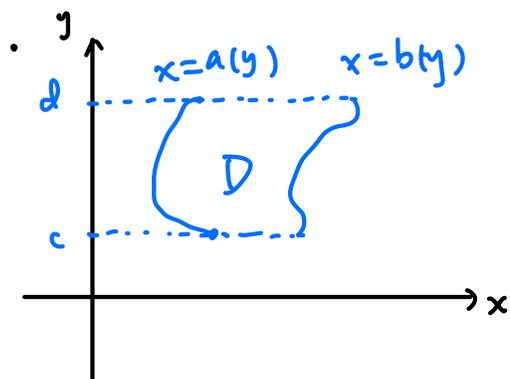
$$\iint_D f(x, y) dA = \iint_R g(x, y) dA$$

• Convert into iterated integral (2 cases)



$$D = \{ (x, y) : a \leq x \leq b, c(x) \leq y \leq d(x) \}$$

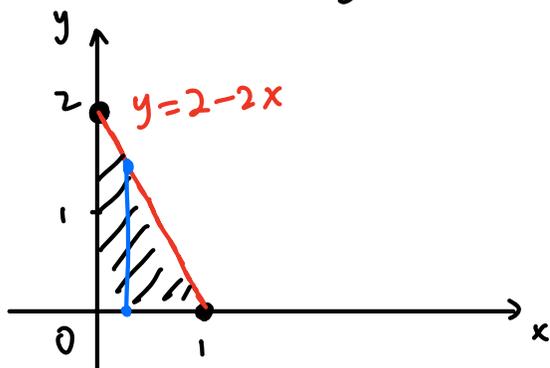
$$\iint_D f(x, y) dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx$$



$$D = \{ (x, y) : c \leq y \leq d, a(y) \leq x \leq b(y) \}$$

$$\iint_D f(x, y) dA = \int_c^d \int_{a(y)}^{b(y)} f(x, y) dx dy$$

Ex Calculate $\iint_D xy dA$, where D is bounded by $x=0$, $y=0$, $y=2-2x$



$$\int_0^1 \int_0^{2-2x} xy dy dx$$

$$= \int_0^1 \left. \frac{1}{2} x y^2 \right|_{y=0}^{2-2x} dx$$

$$= \int_0^1 \frac{1}{2} x (2-2x)^2 dx$$

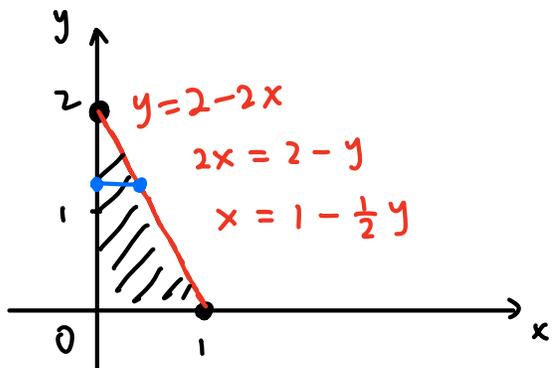
$$= \int_0^1 \frac{1}{2} x (4 - 8x + 4x^2) dx$$

$$= \int_0^1 (2x - 4x^2 + 2x^3) dx$$

$$= \left(2 \cdot \frac{1}{2} x^2 - 4 \cdot \frac{1}{3} x^3 + 2 \cdot \frac{1}{4} x^4 \right) \Big|_{x=0}^1$$

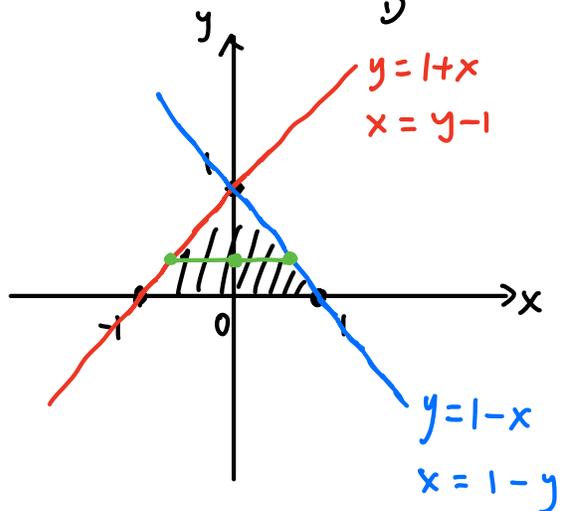
$$= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$$

Another way :



$$\begin{aligned}
 & \int_0^2 \int_0^{1-\frac{1}{2}y} xy \, dx \, dy \\
 &= \int_0^2 \frac{1}{2} x^2 y \Big|_{x=0}^{1-\frac{1}{2}y} dy \\
 &= \int_0^2 \frac{1}{2} \left(1 - \frac{1}{2}y\right)^2 y \, dy \\
 &= \frac{1}{2} \int_0^2 \left(1 - y + \frac{1}{4}y^2\right) y \, dy \\
 &= \frac{1}{2} \int_0^2 \left(y - y^2 + \frac{1}{4}y^3\right) dy \\
 &= \frac{1}{2} \left(\frac{1}{2}y^2 - \frac{1}{3}y^3 + \frac{1}{4} \cdot \frac{1}{4}y^4\right) \Big|_{y=0}^2 \\
 &= \frac{1}{2} \left(\frac{1}{2} \cdot 4 - \frac{1}{3} \cdot 8 + \frac{1}{16} \cdot 16\right) = \frac{1}{6}
 \end{aligned}$$

Ex Calculate $\iint_D (y+1) dA$, where D is bounded by $y=0$, $y=1+x$, $y=1-x$



$$\begin{aligned}
 & \int_0^1 \int_{y-1}^{1-y} (y+1) dx \, dy \\
 &= \int_0^1 (y+1) x \Big|_{x=y-1}^{1-y} dy \\
 &= \int_0^1 (y+1) \left((1-y) - (y-1)\right) dy \\
 &= \int_0^1 (y+1) (2-2y) dy \\
 &= \int_0^1 (2 - 2y^2) dy = \left(2y - 2 \cdot \frac{1}{3}y^3\right) \Big|_{y=0}^1 = \frac{4}{3}
 \end{aligned}$$

Properties of double integrals

• Linear property : $\iint_D (c_1 f(x,y) + c_2 g(x,y)) dA$

$$= c_1 \iint_D f(x,y) dA + c_2 \iint_D g(x,y) dA$$

• If $D = D_1 \cup D_2$ where D_1, D_2 do not overlap except for boundary,

then
$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA$$

Another way for previous example:

$$\iint_D (y+1) dA = \iint_{D_1} (y+1) dA + \iint_{D_2} (y+1) dA$$

$$\iint_{D_1} (y+1) dA = \int_{-1}^0 \int_0^{1+x} (y+1) dy dx$$

$$= \int_{-1}^0 \left(\frac{1}{2} y^2 + y \right) \Big|_{y=0}^{1+x} dx$$

$$= \int_{-1}^0 \left(\frac{1}{2} (1+x)^2 + (1+x) \right) dx$$

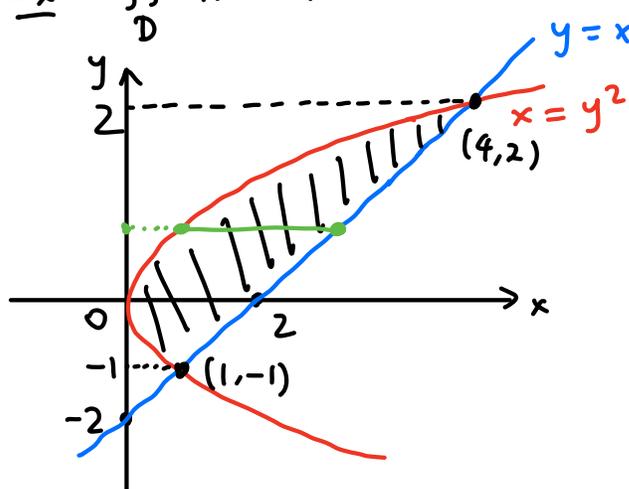
$$= \int_{-1}^0 \left(\frac{1}{2} + x + \frac{1}{2} x^2 + 1 + x \right) dx = \int_{-1}^0 \left(\frac{3}{2} + 2x + \frac{1}{2} x^2 \right) dx$$

$$= \left(\frac{3}{2} x + x^2 + \frac{1}{2} \cdot \frac{1}{3} x^3 \right) \Big|_{x=-1}^0 = - \left(-\frac{3}{2} + 1 - \frac{1}{6} \right) = \frac{2}{3}$$

$$\iint_{D_2} (y+1) dA = \int_0^1 \int_0^{1-x} (y+1) dy dx = \dots = \frac{2}{3}$$

$$\Rightarrow \iint_D (y+1) dA = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

Ex $\iint_D x dA$, where D is bounded by $x = y^2$ and $y = x - 2$



$$\int_{-1}^2 \int_{y^2}^{y+2} x dx dy$$

$$= \int_{-1}^2 \frac{1}{2} x^2 \Big|_{x=y^2}^{y+2} dy$$

$$= \frac{1}{2} \int_{-1}^2 \left((y+2)^2 - y^4 \right) dy$$

Calculate intersection pts:

$$\begin{cases} x = y^2 \\ y = x - 2 \end{cases}$$

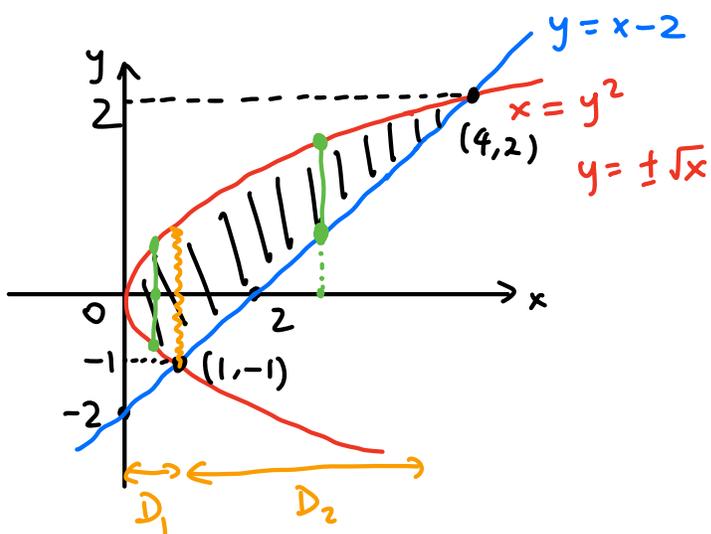
$$y = y^2 - 2 \quad y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y_1 = 2, \quad y_2 = -1$$

$$x_1 = 4, \quad x_2 = 1$$

Another way: x as outer



$$\begin{aligned} &= \frac{1}{2} \left(\frac{1}{3}(y+2)^3 - \frac{1}{5}y^5 \right) \Big|_{y=-1}^2 \\ &= \frac{1}{2} \left[\left(\frac{1}{3} \cdot 4^3 - \frac{1}{5} \cdot 2^5 \right) - \left(\frac{1}{3} + \frac{1}{5} \right) \right] = \frac{36}{5} \end{aligned}$$

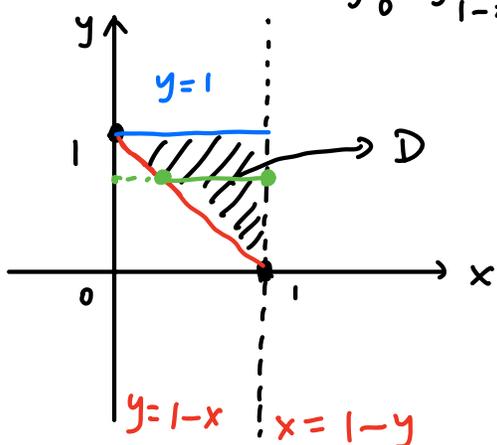
$$\begin{aligned} \iint_{D_1} x \, dA &= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} x \, dy \, dx \\ &= \dots = \frac{4}{5} \end{aligned}$$

$$\begin{aligned} \iint_{D_2} x \, dA &= \int_1^4 \int_{x-2}^{\sqrt{x}} x \, dy \, dx \\ &= \dots = \frac{32}{5} \end{aligned}$$

• Change the order of iterated integral

$$\iint \dots \, dy \, dx \longrightarrow \iint_D \dots \, dA \longrightarrow \iint \dots \, dx \, dy$$

Ex Calculate $\int_0^1 \int_{1-x}^1 e^{y^2} \, dy \, dx$



$$\begin{aligned} &= \iint_D e^{y^2} \, dA = \int_0^1 \int_{1-y}^1 e^{y^2} \, dx \, dy \\ &= \int_0^1 e^{y^2} x \Big|_{x=1-y}^1 \, dy \\ &= \int_0^1 e^{y^2} (1 - (1-y)) \, dy \end{aligned}$$

$$= \int_0^1 e^{y^2} y dy = \frac{1}{2} e^{y^2} \Big|_{y=0}^1 = \frac{1}{2}(e-1)$$

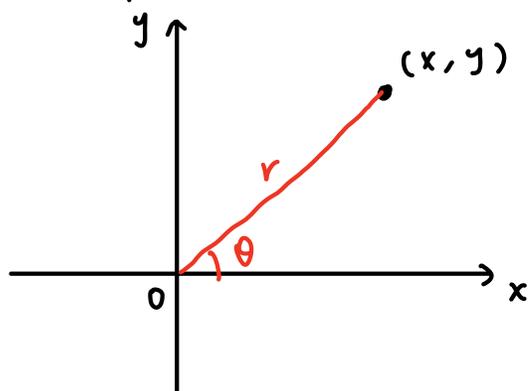
$$\int e^{y^2} y dy = \frac{1}{2} \int e^u du$$

$$u = y^2 \quad = \frac{1}{2} e^u + C$$

$$du = 2y dy \quad = \frac{1}{2} e^{y^2} + C$$

5.3 Double integrals in polar coordinates

Recall polar coordinates:



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Thm If a region D (in (x, y)) can be described by " (r, θ) in R " for some region R , then

$$\iint_D f(x, y) dx dy = \iint_R f(r \cos \theta, r \sin \theta) \underbrace{r}_{\uparrow} dr d\theta$$

pay attention! "Jacobian"

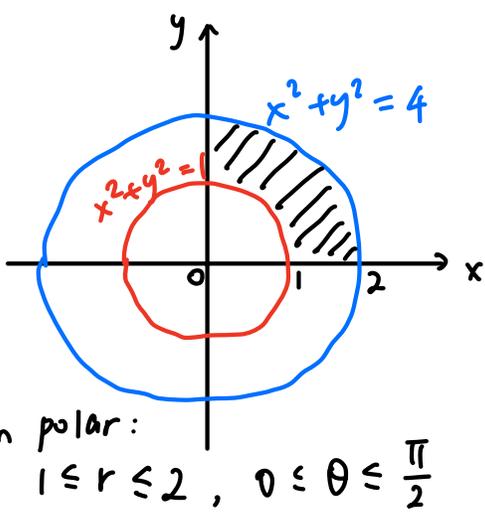
Compare w/ u-sub:

$$\int_a^b f(x) dx = \int_{\dots}^{\dots} f(g(u)) \underbrace{g'(u)} du$$

$$x = g(u)$$

$$dx = g'(u) du$$

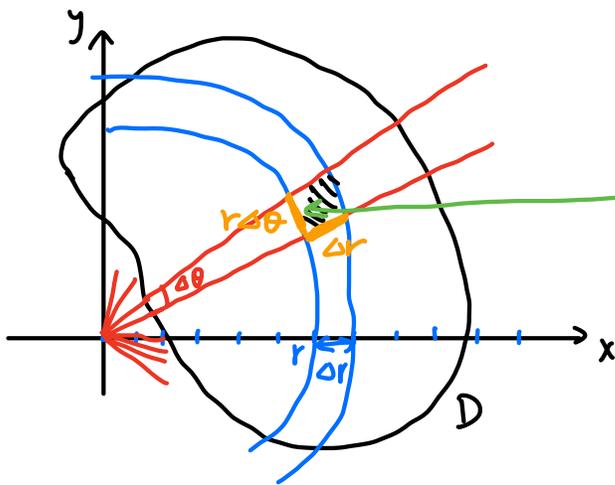
Ex Let D be the region between $x^2 + y^2 = 1$, $x^2 + y^2 = 4$ in the first quadrant. Calculate $\iint_D y dA$.



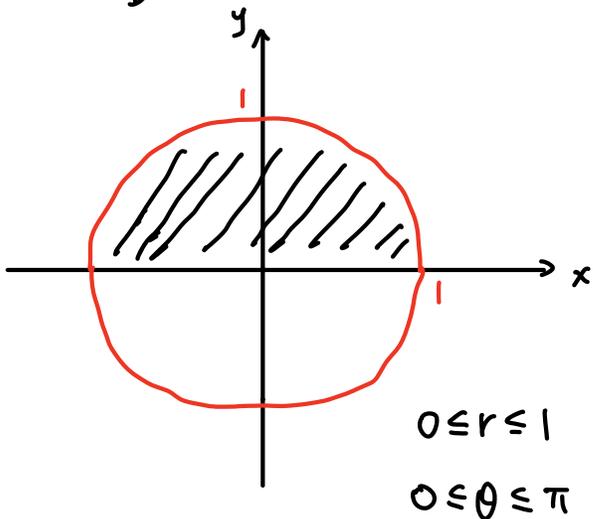
$$\begin{aligned} &= \int_1^2 \int_0^{\frac{\pi}{2}} r \sin \theta \cdot r \, d\theta \, dr \\ &= \int_1^2 r^2 \cdot (-\cos \theta) \Big|_{\theta=0}^{\frac{\pi}{2}} \, dr \\ &= \int_1^2 r^2 \cdot 1 \, dr = \frac{1}{3} r^3 \Big|_{r=1}^2 = \frac{7}{3} \end{aligned}$$

$$\iint_D f(x,y) \, dx \, dy = \iint_R f(r \cos \theta, r \sin \theta) \underbrace{r \, dr \, d\theta}$$

Why an extra r factor?



Ex $\iint_D x^2 y \, dA$, where D is the region inside $x^2 + y^2 = 1$, above x -axis



polar:

$$\begin{aligned} &\int_0^1 \int_0^{\pi} \underbrace{(r \cos \theta)^2 (r \sin \theta) r \, d\theta \, dr}_{r^4 \cos^2 \theta \sin \theta} \\ &= \int_0^1 r^4 \, dr \cdot \int_0^{\pi} \cos^2 \theta \sin \theta \, d\theta \\ &= \frac{1}{5} r^5 \Big|_0^1 \cdot \left(-\frac{1}{3} \cos^3 \theta \right) \Big|_0^{\pi} \\ &= \frac{1}{5} \cdot 1 \cdot \left(-\frac{1}{3} \right) \cdot (-1 - 1) = \boxed{\frac{2}{15}} \end{aligned}$$

"Separation trick":

constant upper/lower bounds
product-form integrand

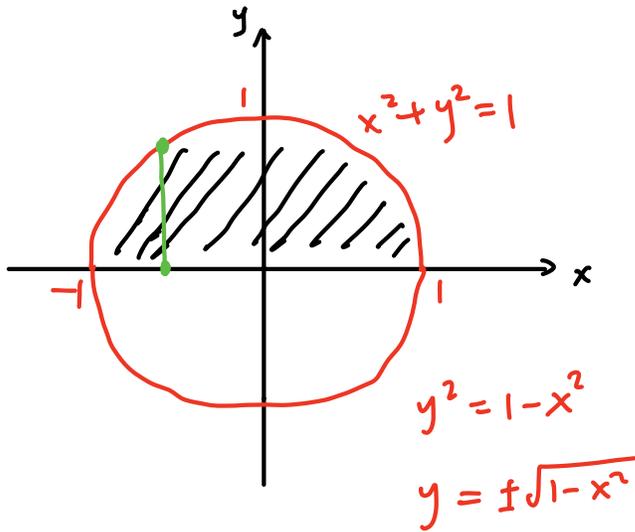
$$\int \cos^2 \theta \sin \theta d\theta = -\int u^2 du$$

$$= -\frac{1}{3} u^3 + C = -\frac{1}{3} \cos^3 \theta + C$$

$$u = \cos \theta$$

$$du = -\sin \theta d\theta$$

Another way: integrate in x, y x as outer



$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x^2 y dy dx$$

$$= \int_{-1}^1 \frac{1}{2} x^2 y^2 \Big|_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_{-1}^1 x^2 (1 - x^2) dx$$

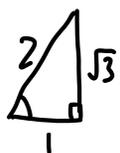
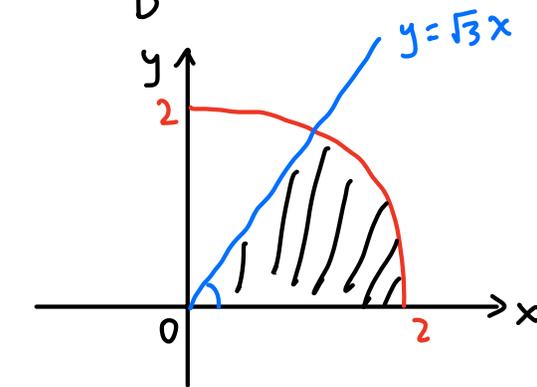
$$= \frac{1}{2} \int_{-1}^1 (x^2 - x^4) dx$$

$$= \frac{1}{2} \left(\frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_{-1}^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) - \left(-\frac{1}{3} + \frac{1}{5} \right) \right] = \frac{2}{15}$$

$$\frac{2}{15} - \left(-\frac{2}{15} \right)$$

Ex $\iint_D x dA$, D is the region inside $x^2 + y^2 = 4$, in first quadrant, below $y = \sqrt{3}x$



$$0 \leq r \leq 2$$

$$0 \leq \theta \leq \frac{\pi}{3}$$

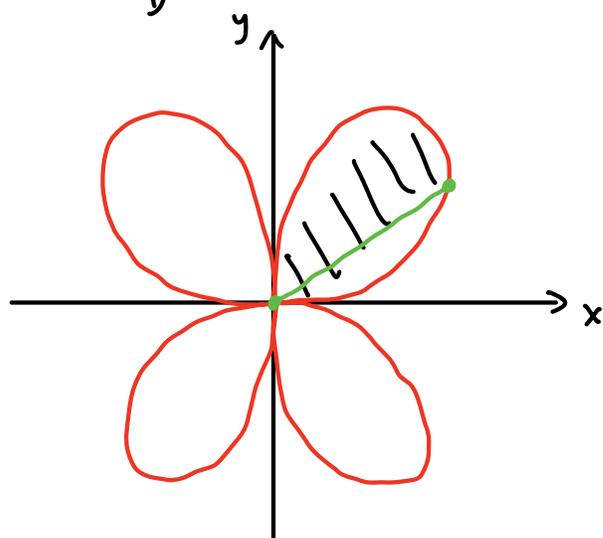
$$\int_0^2 \int_0^{\frac{\pi}{3}} \underbrace{r \cos \theta \cdot r}_{r^2 \cos \theta} d\theta dr$$

$$= \int_0^2 r^2 dr \cdot \int_0^{\frac{\pi}{3}} \cos \theta d\theta$$

$$= \frac{1}{3} r^3 \Big|_0^2 \cdot \sin \theta \Big|_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3} \cdot 8 \cdot \frac{\sqrt{3}}{2} = \frac{4\sqrt{3}}{3}$$

Ex $\iint_D \sqrt{x^2+y^2} dA$, D is the region inside $r = \sin(2\theta)$ in the first quadrant



$$0 \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq \sin(2\theta)$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} r \cdot r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3 \Big|_{r=0}^{\sin(2\theta)} d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin^3(2\theta) d\theta$$

$$= \frac{1}{3} \cdot \left(-\frac{1}{2}\right) \left(\cos(2\theta) - \frac{1}{3} \cos^3(2\theta)\right) \Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{6} \cdot \left[\left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right) \right] = \boxed{\frac{2}{9}}$$

$$\int \sin^3(2\theta) d\theta = \int \sin^2(2\theta) \cdot \sin(2\theta) d\theta$$

$$= \int (1 - \cos^2(2\theta)) \cdot \sin(2\theta) d\theta$$

$$= -\frac{1}{2} \int (1 - u^2) du$$

$$= -\frac{1}{2} \left(u - \frac{1}{3} u^3\right) + C = -\frac{1}{2} \left(\cos(2\theta) - \frac{1}{3} \cos^3(2\theta)\right) + C$$

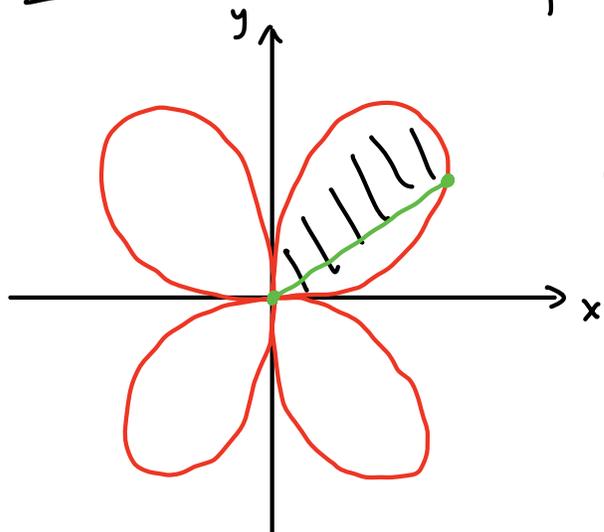
$$u = \cos(2\theta)$$

$$du = -2 \sin(2\theta) d\theta$$

Applications of double integrals (sec. 5.2, 5.3)

• Area of a region D : $\iint_D 1 dA$

Ex Area of region in previous Ex.



$$0 \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq r \leq \sin(2\theta)$$

$$\iint_D 1 dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sin(2\theta)} 1 \cdot r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_{r=0}^{\sin(2\theta)} d\theta$$

$$\cos^2 x = \frac{1}{2} (1 + \cos(2x))$$

$$\sin^2 x = \frac{1}{2} (1 - \cos(2x))$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^2(2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos(4\theta)) d\theta$$

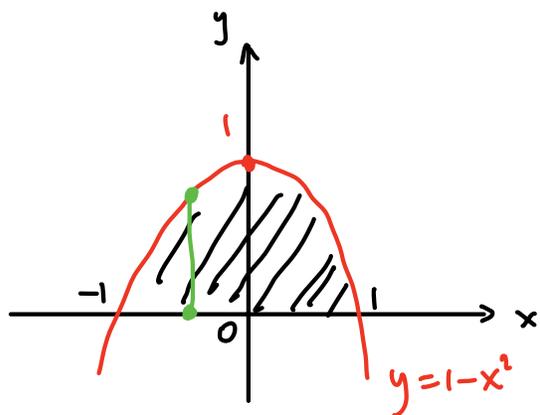
$$= \frac{1}{4} \left(\theta - \frac{1}{4} \sin(4\theta) \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$$

• Average value of $f(x, y)$ on D : defined as

$$f_{\text{ave}} = \frac{1}{\text{Area}(D)} \iint_D f(x, y) dA$$

Ex Find average value of $f(x, y) = y$ over D , where D is bounded by $y = 1 - x^2$, $y = 0$



$$\iint_D y dA$$

$$= \int_{-1}^1 \int_0^{1-x^2} y dy dx$$

$$= \int_{-1}^1 \left. \frac{1}{2} y^2 \right|_{y=0}^{1-x^2} dx$$

$$= \frac{1}{2} \int_{-1}^1 (1-x^2)^2 dx$$

$$= \frac{1}{2} \int_{-1}^1 (1 - 2x^2 + x^4) dx$$

$$= \frac{1}{2} \left(x - 2 \cdot \frac{1}{3} x^3 + \frac{1}{5} x^5 \right) \Big|_{-1}^1$$

$$= \frac{1}{2} \left[\underbrace{\left(1 - \frac{2}{3} + \frac{1}{5} \right)}_{\frac{8}{15}} - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right] = \frac{8}{15}$$

$$\Rightarrow \text{average value} = \frac{\frac{8}{15}}{\frac{4}{3}} = \boxed{\frac{2}{5}}$$

$$\iint_D 1 dA$$

$$= \int_{-1}^1 \int_0^{1-x^2} 1 dy dx$$

$$= \int_{-1}^1 y \Big|_{y=0}^{1-x^2} dx$$

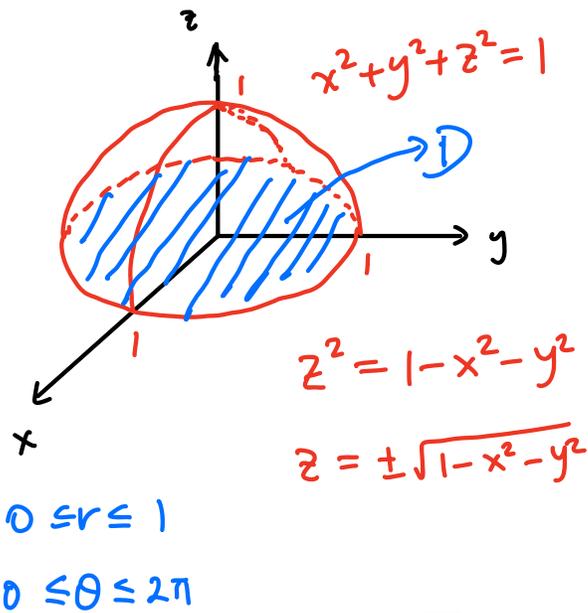
$$= \int_{-1}^1 (1-x^2) dx$$

$$= \left(x - \frac{1}{3} x^3 \right) \Big|_{-1}^1$$

$$= \left(1 - \frac{1}{3} \right) - \left(-1 + \frac{1}{3} \right) = \frac{4}{3}$$

• Find volume (by definition of double integral)

Ex Find volume of unit sphere



volume of unit sphere

$$= 2 \cdot \iint_D \sqrt{1 - x^2 - y^2} dA$$

$$= 2 \int_0^1 \int_0^{2\pi} \sqrt{1 - r^2} r d\theta dr$$

$$= 2 \int_0^1 \sqrt{1 - r^2} r dr \cdot \int_0^{2\pi} 1 d\theta$$

$$= 2 \cdot \left(-\frac{1}{3}\right) (1 - r^2)^{\frac{3}{2}} \Big|_0^1 \cdot 2\pi = \boxed{\frac{4\pi}{3}}$$

$$\int \sqrt{1 - r^2} r dr$$

$$u = 1 - r^2$$

$$du = -2r dr$$

$$= -\frac{1}{2} \int \sqrt{u} du$$

$$= -\frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -\frac{1}{3} (1 - r^2)^{\frac{3}{2}} + C$$