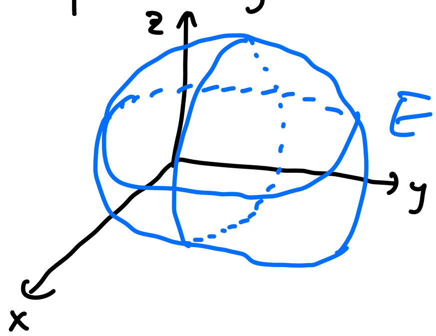
5.4 Triple integrals

Let E be a 3-D region. Define the triple integral



$$\iiint_E f(x,y,z)\,dV$$

similar to how we define double integrals

- When $E = \underset{x}{[a,b]} \times \underset{y}{[c,d]} \times \underset{z}{[e,g]}$

$$\iiint_E f(x,y,z)\,dV = \int_e^g \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

(or any order of x, y, z)

<u>Ex</u> Calculate $\iiint_E xz\,dV$, where $E = \underset{x}{[0,1]} \times \underset{y}{[0,3]} \times \underset{z}{[0,2]}$

$$\int_0^2 \int_0^3 \int_0^1 xz \quad dx\,dy\,dz$$

$$= \int_0^2 \int_0^3 \frac{1}{2}x^2 z \Big|_{x=0}^{1} dy\,dz$$

$$= \int_0^2 \int_0^3 \frac{1}{2} z \,dy\,dz$$

$$= \int_0^2 \frac{1}{2} z y \Big|_{y=0}^{3} dz$$

$$= \int_0^2 \frac{3}{2} z\,dz = \frac{3}{2} \cdot \frac{1}{2} z^2 \Big|_0^2 = 3$$

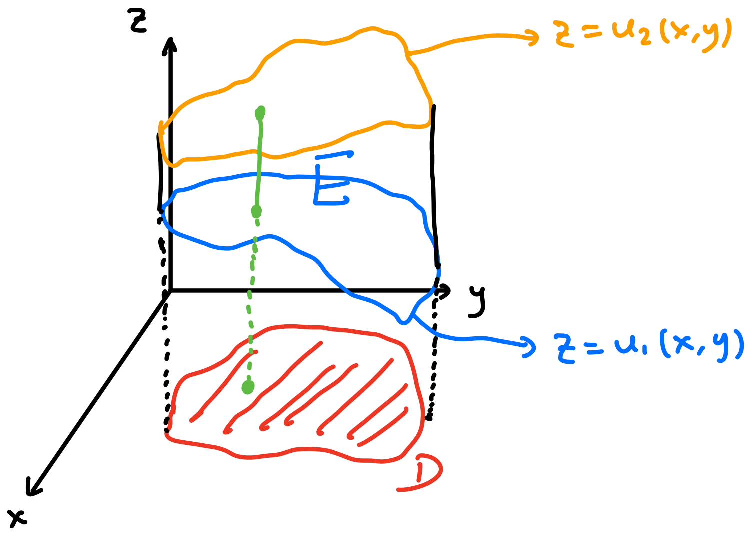
a quicker way:

$$= \int_0^1 x\,dx \cdot \int_0^3 1 \cdot dy \cdot \int_0^2 z\,dz$$

$$= \frac{1}{2}x^2 \Big|_0^1 \cdot 3 \cdot \frac{1}{2} z^2 \Big|_0^2$$

$$= \frac{1}{2} \cdot 1 \cdot 3 \cdot \frac{1}{2} \cdot 2^2 = 3$$

Write general triple integral into iterated integral

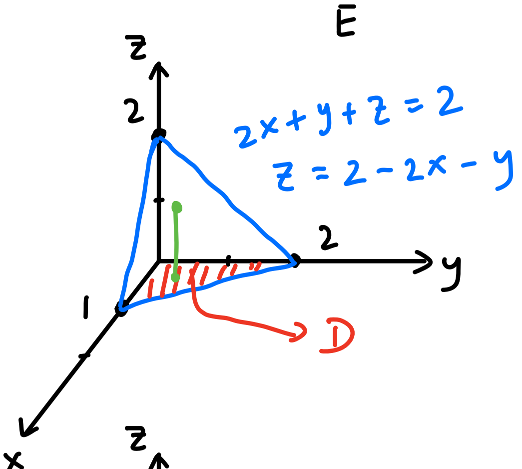


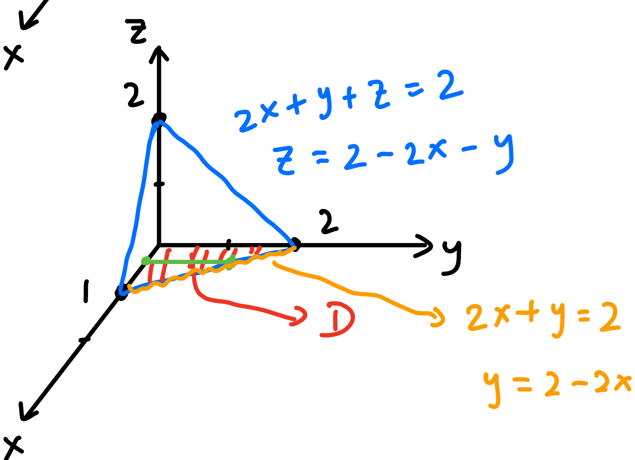
$E = \{ (x,y,z) : (x,y) \text{ in } D, \; u_1(x,y) \le z \le u_2(x,y) \}$.

$$\Rightarrow \iiint_E f(x,y,z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dx\, dy$$

Ex E : in first octant, below the plane $2x + y + z = 2$

Calculate $\iiint_E z \, dV$





$$= \iint_D \left(\int_0^{2-2x-y} z \, dz \right) dx\, dy$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} z \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} \frac{1}{2} z^2 \Big|_{z=0}^{2-2x-y} dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{2-2x} (2-2x-y)^2 \, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{2-2x} (y-2+2x)^2 \, dy\, dx$$

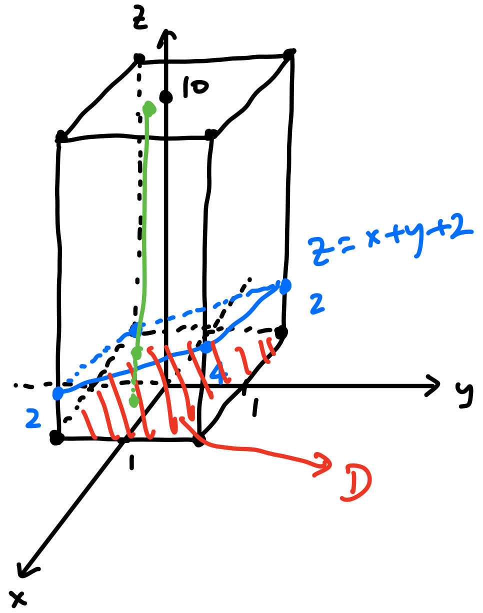
$$= \frac{1}{2} \int_0^1 \frac{1}{3} (y-2+2x)^3 \Big|_{y=0}^{2-2x} dx$$

$$= \frac{1}{6} \int_0^1 -(-2+2x)^3 \, dx$$

$$= -\frac{1}{6} \cdot 2^3 \int_0^1 (x-1)^3 \, dx$$

$$= -\frac{4}{3} \cdot \frac{1}{4} (x-1)^4 \Big|_0^1 = -\frac{1}{3}(0-1) = \frac{1}{3}$$

Ex E : inside the box w/ vertices $(\pm1, \pm1, 0)$, $(\pm1, \pm1, 10)$, above the plane $z = x+y+2$. Calculate $\iiint_E z\, dV$.



$$\iint_D \left(\int_{x+y+2}^{10} z \, dz \right) dx\,dy$$

$$= \int_{-1}^{1} \int_{-1}^{1} \int_{x+y+2}^{10} z \, dz\, dy\, dx$$

$$= \int_{-1}^{1} \int_{-1}^{1} \frac{1}{2} z^2 \Big|_{z=x+y+2}^{10} dy\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \left(100 - (x+y+2)^2\right) dy\, dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left(100y - \frac{1}{3}(y+x+2)^3\right) \Big|_{y=-1}^{1} dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left[\left(100 - \frac{1}{3}(x+3)^3\right) - \left(-100 - \frac{1}{3}(x+1)^3\right) \right] dx$$

$$= \frac{1}{2} \int_{-1}^{1} \left[200 - \frac{1}{3}(x+3)^3 + \frac{1}{3}(x+1)^3 \right] dx$$

$$= \frac{1}{2} \left(200x - \frac{1}{3} \cdot \frac{1}{4}(x+3)^4 + \frac{1}{3} \cdot \frac{1}{4}(x+1)^4 \right) \Big|_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(200 - \frac{1}{12} \cdot 4^4 + \frac{1}{12} \cdot 2^4\right) - \left(-200 - \frac{1}{12} \cdot 2^4\right) \right] = \boxed{\frac{572}{3}}$$