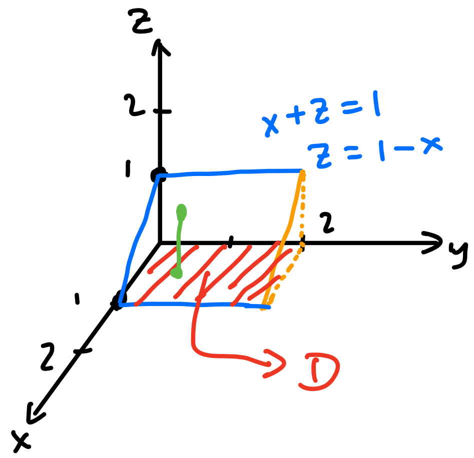
5.4 (continued)

Ex E : in first octant, bounded by $x+z=1$ and $y=2$

Calculate $\iiint_E y e^{-z} \, dV$



$$\iint_D \left(\int_0^{1-x} y e^{-z} dz \right) dx dy$$

$$= \int_0^1 \int_0^2 \int_0^{1-x} y e^{-z} dz \, dy \, dx$$

$$= \int_0^1 \int_0^2 \left(-y e^{-z} \right) \Big|_{z=0}^{1-x} dy \, dx$$

$$= -\int_0^1 \int_0^2 y \left(e^{x-1} - 1 \right) dy \, dx$$

$$= -\int_0^1 \left(e^{x-1} - 1 \right) dx \cdot \int_0^2 y \, dy$$

$$= -\left(e^{x-1} - x \right) \Big|_0^1 \cdot \frac{1}{2} y^2 \Big|_0^2$$

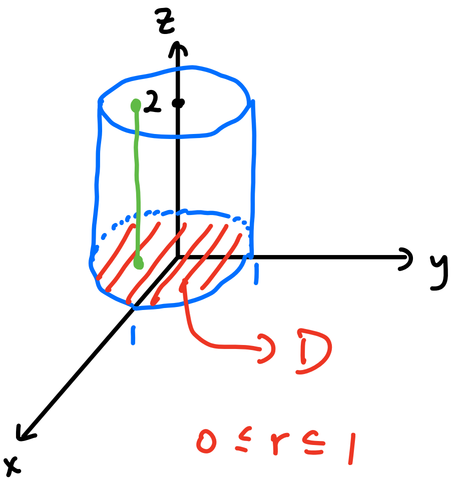
$$= -\left[(1-1) - (e^{-1} - 0) \right] \cdot \frac{1}{2} \cdot 2^2 = 2 e^{-1}$$

5.5 Triple integrals in cylindrical and spherical coordinates

Cylindrical : keep z, use polar for (x, y)

Ex E : in cylinder $x^2+y^2=1$, between $z=0$ and $z=2$.

Calculate $\iiint_E x^2 z \, dV$



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\iint_D \left(\int_0^2 x^2 z \, dz \right) dx\,dy$$

$$= \int_0^1 \int_0^{2\pi} \int_0^2 (r\cos\theta)^2 z \, dz \, r \, d\theta \, dr$$

$$= \int_0^1 \int_0^{2\pi} \int_0^2 r^3 \cos^2\theta \, z \, dz \, d\theta \, dr$$

$$= \underbrace{\int_0^1 r^3 dr}_{\frac{1}{4}r^4\Big|_0^1 = \frac{1}{4}} \cdot \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{\pi} \cdot \underbrace{\int_0^2 z \, dz}_{\frac{1}{2}z^2\Big|_0^2 = 2} = \boxed{\frac{\pi}{2}}$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos(2\theta)) \, d\theta$$

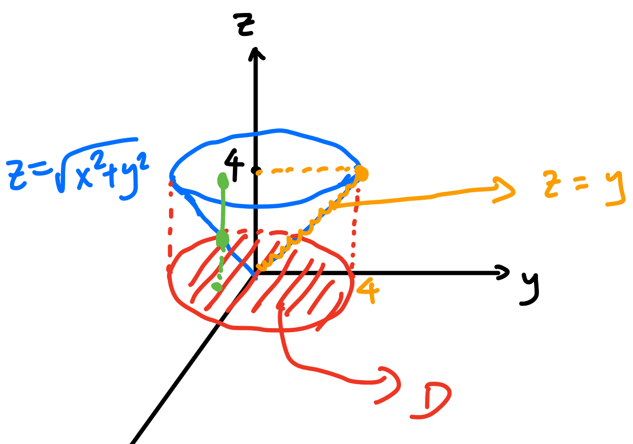
$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2} \cdot 2\pi = \pi$$

SHAPES you need to know (3-D)

plane, cylinder, sphere, cone, paraboloid

Ex E : above the cone $z=\sqrt{x^2+y^2}$, below $z=4$. Calculate $\iiint_E z^2 dV$



$0 \le r \le 4$

$0 \le \theta \le 2\pi$

$$\iint_D \left(\int_{\sqrt{x^2+y^2}}^4 z^2 \, dz \right) dx\,dy$$

$$= \int_0^4 \int_0^{2\pi} \int_r^4 z^2 \, dz \; r \, d\theta \, dr$$

$$= \int_0^4 \int_0^{2\pi} \frac{1}{3} z^3 \Big|_{z=r}^4 \; r \, d\theta \, dr$$

$$= \frac{1}{3} \int_0^4 \int_0^{2\pi} (64 - r^3) r \, d\theta \, dr$$

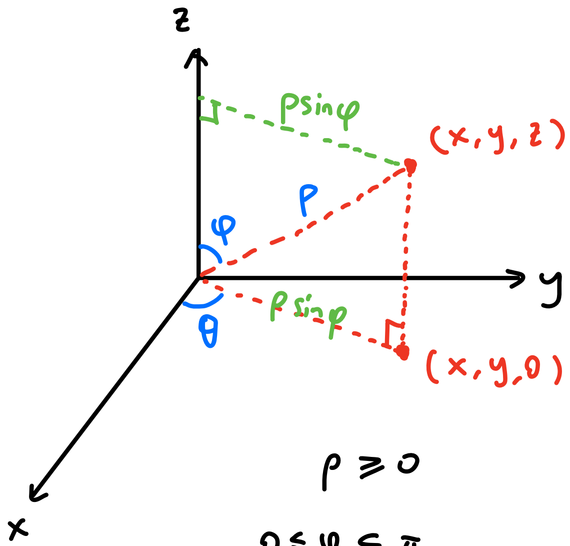
$$= \frac{1}{3} \int_0^4 (64 - r^3) r \, dr \cdot \int_0^{2\pi} 1 \cdot d\theta$$

$$= \frac{1}{3}\int_0^4 (64r - r^4)\,dr \cdot 2\pi$$

$$= \frac{1}{3}\left(64\cdot\frac{1}{2}r^2 - \frac{1}{5}r^5\right)\Big|_0^4 \cdot 2\pi$$

$$= \frac{1}{3}\left(32\times 4^2 - \frac{1}{5}\cdot 4^5\right)\cdot 2\pi = \frac{1024}{5}\pi$$

Spherical coordinates



$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\rho = \sqrt{x^2+y^2+z^2}$$

$\rho \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

Thm Let E be a region (in (x,y,z)) described by "(ρ,φ,θ) in R".
Then

$$\iiint_E f(x,y,z)\,dx\,dy\,dz = \iiint_R f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Ex Calculate $\iiint_E 1\,dV$, E: inside unit sphere.
$x^2+y^2+z^2=1$

$0 \leq \rho \leq 1$

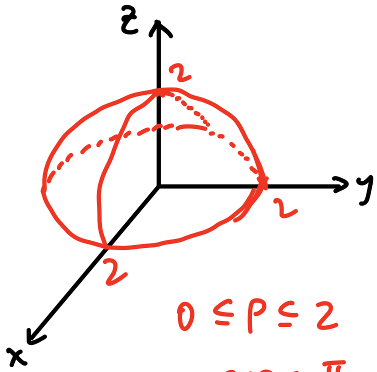
$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} 1\cdot \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \underbrace{\int_0^{2\pi} 1\,d\theta}_{2\pi} \cdot \underbrace{\int_0^{\pi}\sin\varphi\,d\varphi}_{(-\cos\varphi)\big|_0^{\pi} = 2} \cdot \underbrace{\int_0^1 \rho^2\,d\rho}_{\frac{1}{3}\rho^3\big|_0^1 = \frac{1}{3}} = \frac{4\pi}{3}.$$

Ex E: in $x^2+y^2+z^2=4$, w/ $z \ge 0$. Calculate $\iiint_E x^2\,dV$



$0 \le \rho \le 2$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$$\int_0^2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \underbrace{(\rho \sin\varphi \cos\theta)^2 \rho^2 \sin\varphi}_{\rho^4 \sin^3\varphi \cos^2\theta} \, d\theta \, d\varphi \, d\rho$$

$$= \underbrace{\int_0^2 \rho^4 \, d\rho}_{\frac{1}{5}\rho^5\Big|_0^2 = \frac{32}{5}} \cdot \underbrace{\int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi}_{} \cdot \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{\pi} = \boxed{\frac{64}{15}\pi}$$

$$\left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\right)\Big|_0^{\frac{\pi}{2}} = \frac{2}{3}$$

$$\int \sin^3\varphi \, d\varphi = \int \sin^2\varphi \cdot \sin\varphi \, d\varphi$$

$$= \int (1-\cos^2\varphi)\sin\varphi \, d\varphi$$

$u = \cos\varphi$

$du = -\sin\varphi \, d\varphi$

$$= -\int (1-u^2)\,du$$

$$= -\left(u - \frac{1}{3}u^3\right) + C$$

$$= -\left(\cos\varphi - \frac{1}{3}\cos^3\varphi\right) + C$$