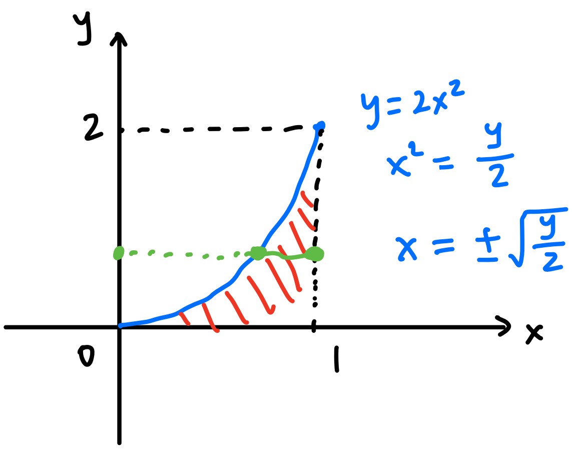
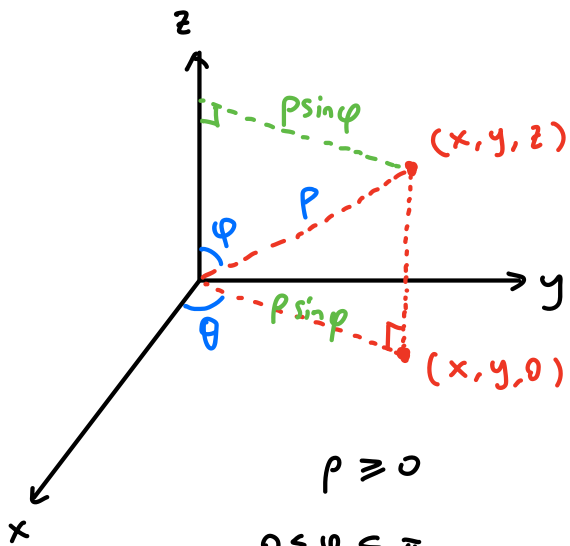
Quiz 5 # 1

$$\int_0^1 \int_0^{2x^2} \sin(y^2)\, dy\, dx = \int_0^2 \int_{\sqrt{\frac{y}{2}}}^1 \sin(y^2)\, dx\, dy$$



Spherical coordinates (recall)



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

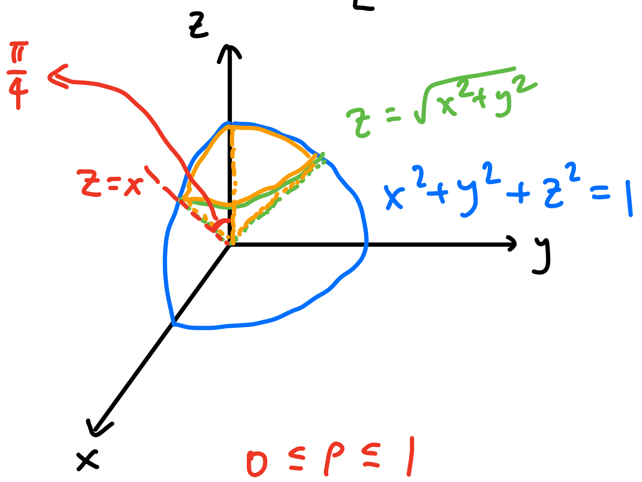
$\rho \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

Ex E : in $x^2+y^2+z^2=1$, in first octant, above cone $z=\sqrt{x^2+y^2}$

Calculate $\iiint_E y \, dV$



$0 \le \rho \le 1$

$0 \le \varphi \le \frac{\pi}{4}$

$0 \le \theta \le \frac{\pi}{2}$

$$\int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \underbrace{\rho \sin\varphi \sin\theta \cdot \rho^2 \sin\varphi}_{\rho^3 \sin^2\varphi \sin\theta} \, d\theta \, d\varphi \, d\rho$$

$$= \underbrace{\int_0^1 \rho^3 \, d\rho}_{\frac{1}{4}\rho^4\Big|_0^1 = \frac{1}{4}} \cdot \underbrace{\int_0^{\frac{\pi}{4}} \sin^2\varphi \, d\varphi}_{\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)} \cdot \underbrace{\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta}_{-\cos\theta\Big|_0^{\frac{\pi}{2}} = 1}$$

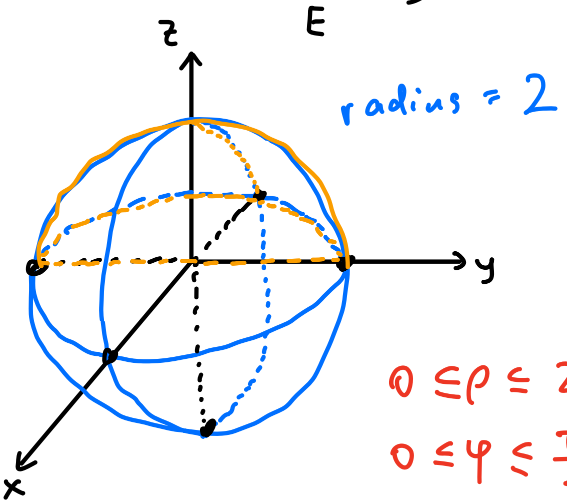
$$= \boxed{\frac{1}{8}\left(\frac{\pi}{4}-\frac{1}{2}\right)}$$

$$\int_0^{\frac{\pi}{4}} \sin^2\varphi \, d\varphi = \int_0^{\frac{\pi}{4}} \frac{1}{2}\left(1-\cos(2\varphi)\right) d\varphi$$

$$= \frac{1}{2}\left(\varphi - \frac{1}{2}\sin(2\varphi)\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

Ex E : inside $x^2+y^2+z^2=4$, satisfying $x \le 0$, $z \ge 0$

Calculate $\iiint_E xy \, dV$



$0 \le \rho \le 2$

$0 \le \varphi \le \frac{\pi}{2}$

$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$

$$\int_0^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \underbrace{\rho \sin\varphi \cos\theta \cdot \rho \sin\varphi \sin\theta \cdot \rho^2 \sin\varphi}_{\rho^4 \sin^3\varphi \cos\theta \sin\theta} \, d\theta \, d\varphi \, d\rho$$

$$= \int_0^2 \rho^4 \, d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^3\varphi \, d\varphi \cdot \underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \sin\theta \, d\theta}_{-\frac{1}{2}\cos^2\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 0}$$

$$= 0.$$

$$\int \cos\theta \sin\theta \, d\theta \qquad u = \cos\theta$$

$$du = -\sin\theta \, d\theta$$

$$= -\int u \, du$$

$$= -\frac{1}{2}u^2 + C$$

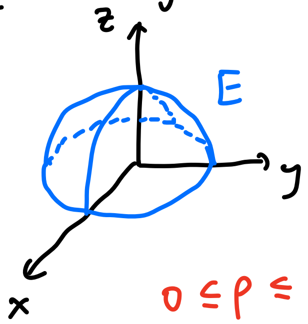
$$= -\frac{1}{2}\cos^2\theta + C$$

- Average value of $f(x,y,z)$ in E :

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\,dV = \frac{\iiint_E f(x,y,z)\,dV}{\iiint_E 1 \, dV}$$

$$V(E) = \iiint_E 1 \, dV \quad \text{volume of } E$$

<u>Ex</u> Average value of $f(x,y,z) = z$ in upper hemisphere of $x^2+y^2+z^2=1$



$0 \le \rho \le 1$

$0 \le \varphi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

$$\iiint_E z\,dV = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \underbrace{\rho\cos\varphi \cdot \rho^2 \sin\varphi}_{\rho^3 \cos\varphi \sin\varphi} \, d\theta \, d\varphi \, d\rho$$

$$= \underbrace{\int_0^1 \rho^3 \, d\rho}_{\frac{1}{4}\rho^4\big|_0^1 = \frac{1}{4}} \cdot \underbrace{\int_0^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi}_{-\frac{1}{2}\cos^2\varphi\big|_0^{\frac{\pi}{2}} = \frac{1}{2}} \cdot \underbrace{\int_0^{2\pi} 1 \, d\theta}_{2\pi} = \frac{\pi}{4}$$

$$\iiint_E 1 \, dV = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 1 \cdot \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$$

$$= \underbrace{\int_0^1 \rho^2 \, d\rho}_{\frac{1}{3}\rho^3\big|_0^1 = \frac{1}{3}} \cdot \underbrace{\int_0^{\frac{\pi}{2}} \sin\varphi \, d\varphi}_{-\cos\varphi\big|_0^{\frac{\pi}{2}} = 1} \cdot \underbrace{\int_0^{2\pi} 1 \, d\theta}_{2\pi} = \frac{2\pi}{3}$$

$$f_{ave} = \frac{\frac{\pi}{4}}{\frac{2\pi}{3}} = \frac{3}{8} .$$

5.6 Center of mass, moments of inertia

Consider a solid object Q w/ density $\rho(x,y,z)$. Its mass is

$$m = \iiint_Q \rho(x,y,z)\,dV$$

Its center of mass is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\iiint_Q x\,\rho(x,y,z)\,dV}{\iiint_Q \rho(x,y,z)\,dV} \qquad \bar{y} = \frac{\iiint_Q y\,\rho(x,y,z)\,dV}{\iiint_Q \rho(x,y,z)\,dV}$$

$$\bar{z} = \frac{\iiint_Q z\,\rho(x,y,z)\,dV}{\iiint_Q \rho(x,y,z)\,dV}$$

"weighted average" of x, y, z

Ex Consider the box $Q = [0,1]\times[0,1]\times[-1,1]$ whose density at (x,y,z) is xz^2. Find center of mass.

$$m = \iiint_Q xz^2\,dV = \int_0^1\int_0^1\int_{-1}^1 xz^2\,dz\,dy\,dx$$

$$= \underbrace{\int_0^1 x\,dx}_{\frac{1}{2}x^2\big|_0^1 = \frac{1}{2}} \cdot \underbrace{\int_0^1 1\,dy}_{1} \cdot \underbrace{\int_{-1}^1 z^2\,dz}_{\frac{1}{3}z^3\big|_{-1}^1 = \frac{2}{3}} = \frac{1}{3}$$

$$\iiint_Q x\cdot xz^2\,dV = \int_0^1\int_0^1\int_{-1}^1 x^2z^2\,dz\,dy\,dx$$

$$= \underbrace{\int_0^1 x^2\,dx}_{\frac{1}{3}x^3\big|_0^1 = \frac{1}{3}} \cdot \underbrace{\int_0^1 1\,dy}_{1} \cdot \underbrace{\int_{-1}^1 z^2\,dz}_{\frac{1}{3}z^3\big|_{-1}^1 = \frac{2}{3}} = \frac{2}{9}$$

$$\Rightarrow \bar{x} = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{2}{3}$$

$$\iiint_Q y \cdot x z^2 \, dV = \int_0^1 \int_0^1 \int_{-1}^1 y x z^2 \, dz \, dy \, dx$$

$$\Rightarrow \bar{y} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

$$= \underbrace{\int_0^1 x \, dx}_{\frac{1}{2}x^2\big|_0^1 = \frac{1}{2}} \cdot \underbrace{\int_0^1 y \, dy}_{\frac{1}{2}y^2\big|_0^1 = \frac{1}{2}} \cdot \underbrace{\int_{-1}^1 z^2 \, dz}_{\frac{1}{3}z^3\big|_{-1}^1 = \frac{2}{3}} = \frac{1}{6}$$

$$\iiint_Q z \cdot x z^2 \, dV = \int_0^1 \int_0^1 \int_{-1}^1 x z^3 \, dz \, dy \, dx$$

$$\Rightarrow \bar{z} = 0$$

$$= \underbrace{\int_0^1 x \, dx}_{\frac{1}{2}x^2\big|_0^1 = \frac{1}{2}} \cdot \underbrace{\int_0^1 1 \, dy}_{1} \cdot \underbrace{\int_{-1}^1 z^3 \, dz}_{\frac{1}{4}z^4\big|_{-1}^1 = 0} = 0$$

$\Rightarrow$ center of mass $\left(\frac{2}{3}, \frac{1}{2}, 0\right)$.