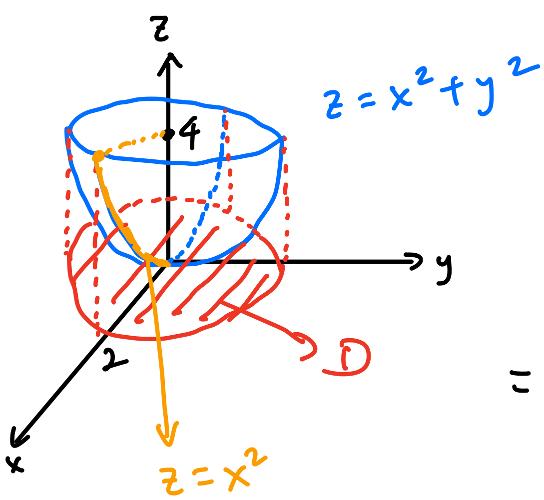
5.6 (continued)

Ex Q : above the paraboloid $z = x^2 + y^2$, below $z = 4$,

density at (x, y, z) is $x^2 + y^2$. Find $\bar{z}$



$$\bar{z} = \frac{\iiint_Q z\,\rho(x,y,z)\,dV}{\iiint_Q \rho(x,y,z)\,dV}$$

$$\iiint_Q z\,\rho(x,y,z)\,dV$$

$$= \iint_D \left(\int_{x^2+y^2}^{4} z\,(x^2+y^2)\,dz \right) dx\,dy$$

$$= \int_0^2 \int_0^{2\pi} \int_{r^2}^{4} z\,r^2\,dz\ r\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} r^2 \tfrac{1}{2} z^2 \Big|_{z=r^2}^{4} r\,d\theta\,dr$$

$$= \frac{1}{2} \int_0^2 \int_0^{2\pi} r^3 (16 - r^4)\,d\theta\,dr$$

$$= \frac{1}{2} \int_0^2 (16r^3 - r^7)\,dr \cdot \int_0^{2\pi} 1\,d\theta$$

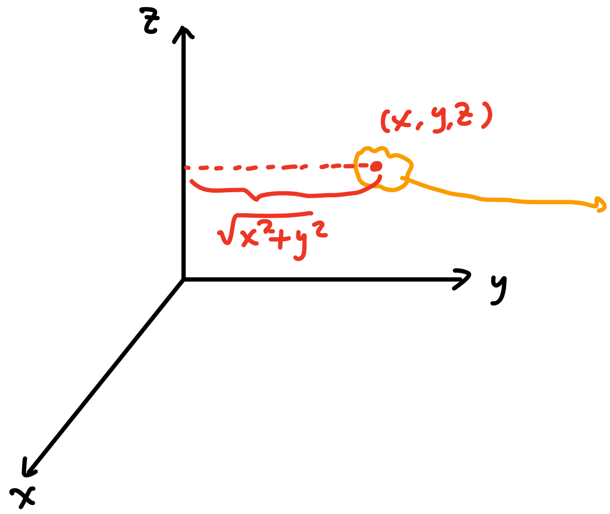
$$= \frac{1}{2} \left(16 \cdot \tfrac{1}{4} r^4 - \tfrac{1}{8} r^8\right)\Big|_0^2 \cdot 2\pi = 32\pi$$

$$\iiint_Q \rho(x,y,z)\,dV = \iint_D \left(\int_{x^2+y^2}^{4} (x^2+y^2)\,dz \right) dx\,dy = \int_0^2 \int_0^{2\pi} \int_{r^2}^{4} r^2\,dz\ r\,d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} r^2 (4 - r^2)\,r\,d\theta\,dr = \int_0^2 \underbrace{r^3(4-r^2)}_{4r^3 - r^5}\,dr \cdot \int_0^{2\pi} 1\,d\theta$$

$$= \left(4 \cdot \tfrac{1}{4} r^4 - \tfrac{1}{6} r^6\right)\Big|_0^2 \cdot 2\pi = \frac{32}{3}\pi \quad \Rightarrow \bar{z} = \frac{32\pi}{\frac{32}{3}\pi} = \boxed{3}$$

- Moments of inertia of a solid Q w/ density $\rho(x,y,z)$ (about coordinate axes)



$$I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV$$

For element ΔV near (x, y, z) moment of inertia for this element

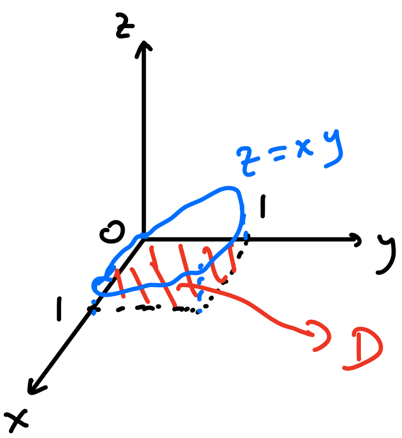
$$\Delta m \cdot \left(\sqrt{x^2+y^2}\right)^2 = \rho(x,y,z)\,\Delta V \cdot (x^2+y^2)$$

$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV$$

$$I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV$$

Ex Q : $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le xy$, density $\rho(x,y,z) = z$

Find I_y



$$I_y = \iiint_Q (x^2+z^2)\,z\,dV$$

$$= \iint_D \left(\int_0^{xy} (x^2 z + z^3)\,dz \right) dx\,dy$$

$$= \int_0^1 \int_0^1 \int_0^{xy} (x^2 z + z^3)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 \left(x^2 \tfrac{1}{2} z^2 + \tfrac{1}{4} z^4 \right) \Big|_{z=0}^{xy} dy\,dx$$

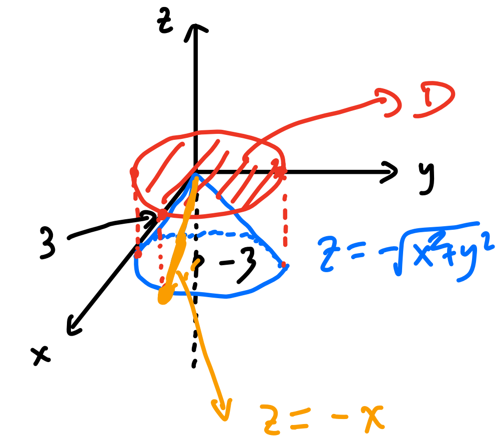
$$= \int_0^1 \int_0^1 \left(\tfrac{1}{2} x^4 y^2 + \tfrac{1}{4} x^4 y^4 \right) dy\,dx$$

$$= \tfrac{1}{2} \int_0^1 x^4 dx \cdot \int_0^1 y^2 dy + \tfrac{1}{4} \int_0^1 x^4 dx \cdot \int_0^1 y^4 dy$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{5} \cdot \tfrac{1}{3} + \tfrac{1}{4} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} = \boxed{\tfrac{13}{300}}$$

Ex Q : below cone $z = -\sqrt{x^2+y^2}$, above $z=-3$, density $\rho(x,y,z) = z^2$

Calculate moment of inertia about z-axis



$$I_z = \iiint_Q (x^2+y^2)\, z^2 \, dV$$

$$= \iint_D \left(\int_{-3}^{-\sqrt{x^2+y^2}} (x^2+y^2) z^2 \, dz \right) dx\,dy$$

$$= \int_0^3 \int_0^{2\pi} \int_{-3}^{-r} r^2 z^2 \, dz \; r \, d\theta \, dr$$

$$= \int_0^3 \int_0^{2\pi} r^2 \tfrac{1}{3} z^3 \Big|_{z=-3}^{-r} r \, d\theta \, dr$$

$$= \frac{1}{3} \int_0^3 \int_0^{2\pi} r^3 (-r^3 + 27) \, d\theta \, dr$$

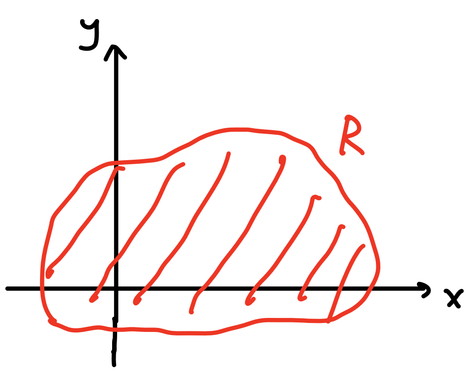
$$= \frac{1}{3} \int_0^3 (-r^6 + 27 r^3) \, dr \cdot \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{1}{3} \left(-\frac{1}{7} r^7 + 27 \cdot \frac{1}{4} r^4 \right)\Big|_0^3 \cdot 2\pi = \boxed{\frac{3^7}{14}\pi}$$

$$-\frac{1}{7}\cdot 3^7 + 3^3 \cdot \frac{1}{4} \cdot 3^4$$

$$3^7 \cdot \frac{3}{28}$$

- mass, center of mass, moments of inertia for a lamina R (thin plate) w/ density $\rho(x,y)$



$$m = \iint_R \rho(x,y) \, dA$$

center of mass $(\bar{x}, \bar{y})$

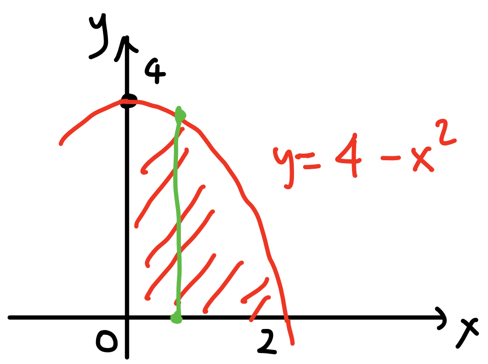
$$\bar{x} = \frac{\iint_R x \rho(x,y) \, dA}{\iint_R \rho(x,y) \, dA} \qquad \bar{y} = \frac{\iint_R y \rho(x,y) \, dA}{\iint_R \rho(x,y) \, dA}$$

moments of inertia

$$I_x = \iint_R y^2 \rho(x,y)\,dA \qquad I_y = \iint_R x^2 \rho(x,y)\,dA$$

$$I_0 = \iint_R (x^2+y^2)\rho(x,y)\,dA = I_x + I_y$$

Ex R : lamina in first quadrant, below $y = 4 - x^2$, density $\rho(x,y) = x$. Calculate $\bar{x}$, I_x



$$\bar{x} = \frac{\iint_R x \cdot x\,dA}{\iint_R x\,dA}$$

$$\iint_R x \cdot x\,dA = \int_0^2 \int_0^{4-x^2} x^2\,dy\,dx$$

$$= \int_0^2 \underbrace{x^2(4-x^2)}_{4x^2 - x^4}\,dx$$

$$= \left(4 \cdot \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^2 = \frac{64}{15}$$

$$\iint_R x\,dA = \int_0^2 \int_0^{4-x^2} x\,dy\,dx = \int_0^2 \underbrace{x(4-x^2)}_{4x - x^3}\,dx$$

$$= \left(4 \cdot \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^2 = 4 \qquad \Rightarrow \quad \bar{x} = \frac{\frac{64}{15}}{4} = \boxed{\frac{16}{15}}$$

$$I_x = \iint_R y^2 x\,dA = \int_0^2 \int_0^{4-x^2} y^2 x\,dy\,dx = \int_0^2 x \cdot \tfrac{1}{3}y^3 \Big|_{y=0}^{4-x^2} dx$$

$$= \frac{1}{3}\int_0^2 x(4-x^2)^3\,dx = \frac{1}{3}\left(-\frac{1}{8}\right)(4-x^2)^4\Big|_0^2 = \frac{1}{24} \cdot 4^4 = \boxed{\frac{32}{3}}$$

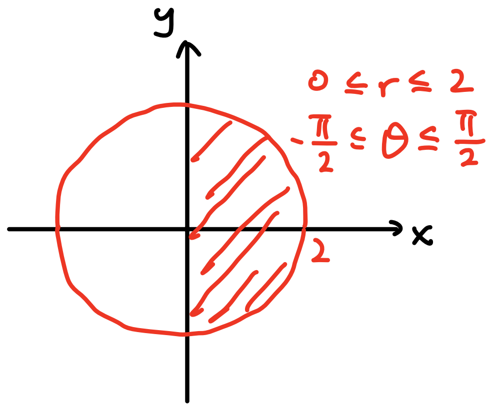
$$\int x(4-x^2)^3\,dx = -\frac{1}{2}\int u^3\,du$$

$u = 4 - x^2$ $\qquad = -\frac{1}{2} \cdot \frac{1}{4}u^4 + C$

$du = -2x\,dx$ $\qquad = -\frac{1}{8}(4-x^2)^4 + C$

$\underline{\text{Ex}}$ R : lamina inside $x^2+y^2=4$, w/ $x \geq 0$, density $\rho(x,y)=\sqrt{x^2+y^2}$

Calculate I_y



$$I_y = \iint_R x^2 \sqrt{x^2+y^2}\, dA$$

$$= \int_0^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{r^2\cos^2\theta \cdot r \cdot r}_{r^4\cos^2\theta}\, d\theta\, dr$$

$$= \underbrace{\int_0^2 r^4\, dr}_{\frac{1}{5}r^5\Big|_0^2 = \frac{32}{5}} \cdot \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta}_{\frac{\pi}{2}} = \boxed{\frac{16}{5}\pi}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(1+\cos(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$$