6.1 Vector fields

number $\mapsto$ number — function $f(x)$

number $\mapsto$ vector — vector-valued function $\vec{r}(t)$

vector $\mapsto$ number — multi-variable function $f(x,y)$

vector $\mapsto$ vector — vector field

Def A vector field $\vec{F}(x,y)$ in $\mathbb{R}^2$ maps a point (x,y) in D (a subset of $\mathbb{R}^2$) to a two-dimensional vector.

domain of $\vec{F}$

Similar define vector fields in $\mathbb{R}^3$.

Examples :

2D vector fields :

$$\vec{F}(x,y) = \langle y, -x \rangle \qquad \vec{G}(x,y) = \langle x^2+y^2, 2xy \rangle$$

3D vector fields :

$$\vec{H}(x,y,z) = \langle x^2 y, -z+x, e^{xyz} \rangle$$

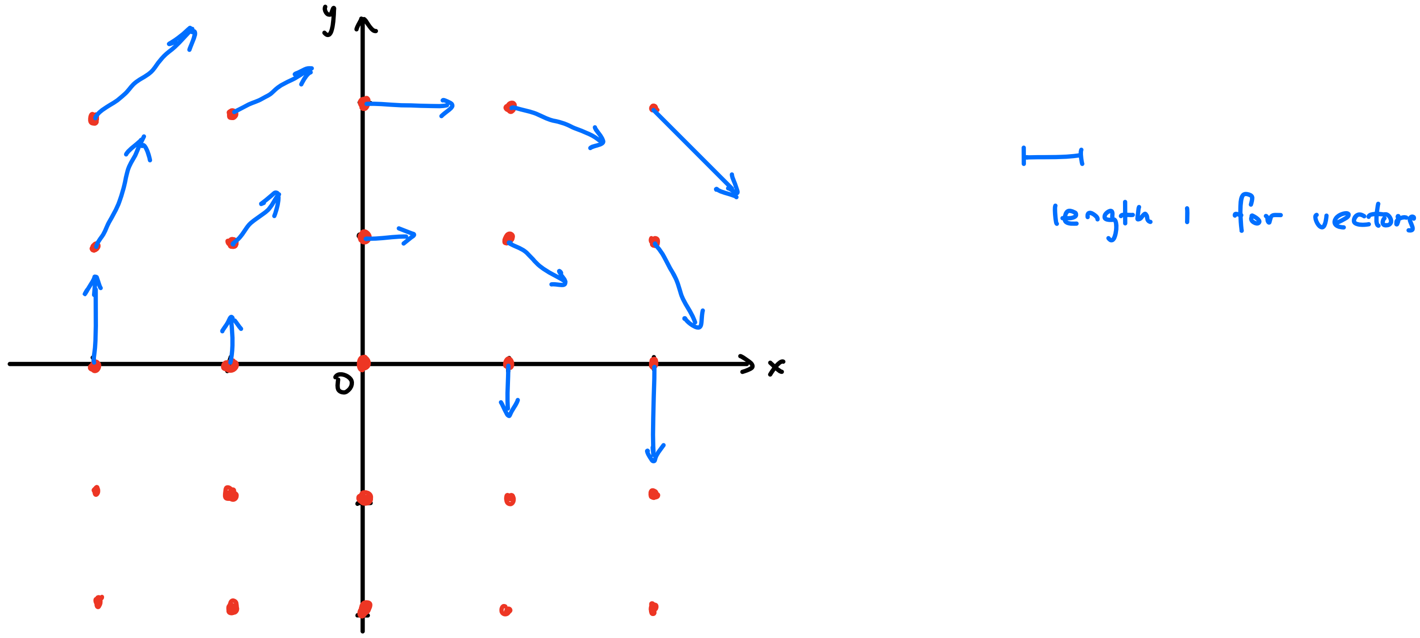
- Realistic examples of vector fields:

  electric field, force field, magnetic field,

  fluid velocity field.

- Draw 2D vector field: pick many pts in $\mathbb{R}^2$. For each pt (x,y), draw $\vec{F}(x,y)$ w/ base pt (x,y)

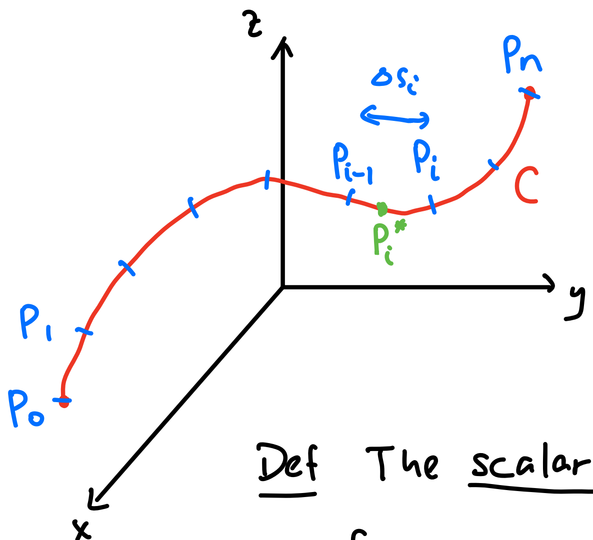
For example, $\vec{F}(x,y) = \langle y, -x \rangle$



6.2 Line integrals

- Scalar line integrals

Motivating example: mass of a wire, given by a curve C, w/ density $f(x,y,z)$



mass of arc P_{i-1} to P_i: approximately

$$f(P_i^*)\underbrace{\Delta s_i}_{\text{length of } i\text{-th arc}}$$

$\Rightarrow$ total mass $\approx \sum_{i=1}^{n} f(P_i^*)\Delta s_i$

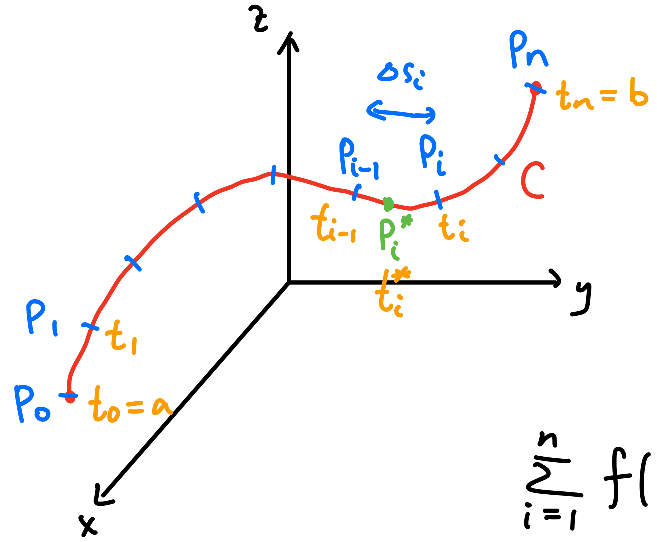
Take limit as partition gets finer.

<u>Def</u> The <u>scalar line integral</u> of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i$$

- Calculate scalar line integral by parametrization

$C: \vec{r}(t), \quad a \le t \le b$



$$\Delta s_i = \int_{t_{i-1}}^{t_i} \|\vec{r}\,'(t)\|\,dt$$

$$\approx \|\vec{r}\,'(t_i^*)\|\,\Delta t_i$$

$$\Delta t_i = t_i - t_{i-1}$$

$$\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \approx \sum_{i=1}^{n} f(\vec{r}(t_i^*))\|\vec{r}\,'(t_i^*)\|\,\Delta t_i$$

Take $n \to \infty$,

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

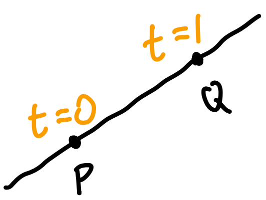
" $ds = \|\vec{r}\,'(t)\|\,dt$ "

<u>Ex</u> Let C be the line segment connecting $(1,0,0)$ and $(0,2,1)$

P $\quad$ Q

Calculate $\int_C xy\,ds$

$\vec{PQ} = \langle -1, 2, 1 \rangle$



$C: \quad \vec{r}(t) = \langle 1,0,0 \rangle + t\langle -1,2,1 \rangle = \langle 1-t, 2t, t \rangle$

$\quad 0 \le t \le 1$

$x \quad y \quad z$

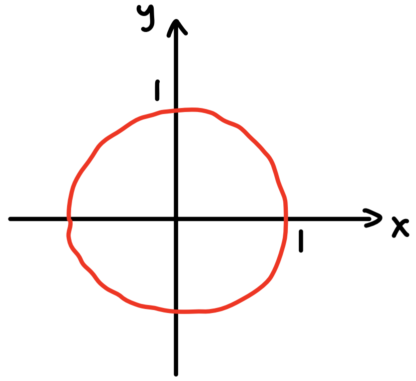
$$\vec{r}\,'(t) = \langle -1, 2, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\int_C xy\,ds = \int_0^1 (1-t)\cdot 2t \cdot \sqrt{6}\,dt = 2\sqrt{6}\int_0^1 (t - t^2)\,dt$$

$$= 2\sqrt{6}\left(\tfrac{1}{2}t^2 - \tfrac{1}{3}t^3\right)\Big|_0^1 = 2\sqrt{6}\cdot\frac{1}{6} = \boxed{\frac{\sqrt{6}}{3}}$$

Ex Let C be the unit circle in $\mathbb{R}^2$. Calculate $\int_C x^2 \, ds$



parametrize

$$x^2 + y^2 = R^2$$

$$\vec{r}(t) = \langle R\cos t, R\sin t \rangle$$

$$C: \quad \vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle \qquad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

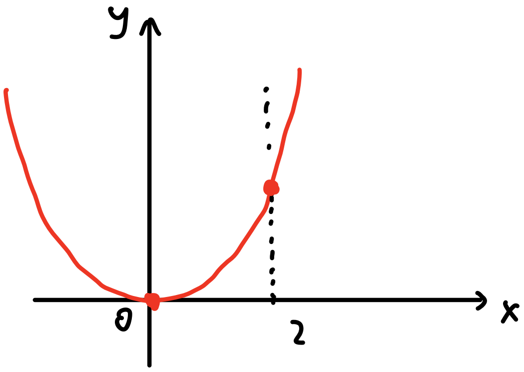
$$\| \vec{r}\,'(t) \| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\int_C x^2 \, ds = \int_0^{2\pi} \cos^2 t \cdot 1 \, dt$$

$$= \int_0^{2\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_0^{2\pi} = \frac{1}{2} \cdot 2\pi = \boxed{\pi}$$

Ex Let C be the part of the parabola $y = x^2$ between $x = 0$ and $x = 2$. Calculate $\int_C \sqrt{1+4y} \, ds$



$$C: \begin{cases} x = t \\ y = t^2 \end{cases} \qquad 0 \le t \le 2$$

$$\vec{r}(t) = \langle t, t^2 \rangle$$

$$\vec{r}\,'(t) = \langle 1, 2t \rangle$$

$$\| \vec{r}\,'(t) \| = \sqrt{1 + 4t^2}$$

$$\int_C \sqrt{1+4y} \, ds = \int_0^2 \sqrt{1 + 4 \cdot t^2} \cdot \sqrt{1 + 4t^2} \, dt = \int_0^2 (1 + 4t^2) \, dt$$

$$= \left(t + \frac{4}{3}t^3\right)\Big|_0^2 = 2 + \frac{4}{3} \cdot 2^3 = \boxed{\frac{38}{3}}$$

x y z

<u>Ex</u> A wire is given by the curve $C: \vec{r}(t) = \langle \cos t, \sin t, t \rangle$
$0 \le t \le 10\pi$

density $f(x,y,z) = z$. Calculate mass.

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\int_C z\, ds = \int_0^{10\pi} t \cdot \sqrt{2}\, dt = \sqrt{2} \cdot \frac{1}{2} t^2 \Big|_0^{10\pi} = \frac{\sqrt{2}}{2}(10\pi)^2 = \boxed{50\sqrt{2}\,\pi^2}$$