Math 2500 | Formula sheet for midterm 1

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Section 2.5:

Vector equation for a line: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{v}$ is a direction vector, and $\mathbf{r}_0$ is the position vector of a point on the line.

Equation for a plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, where $\langle a, b, c \rangle$ is a normal vector, and (x_0, y_0, z_0) is a point on the plane.

Distance from a point P to a line: $\frac{\|\overrightarrow{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$, where $\mathbf{v}$ is a direction vector of the line, and M is a point on the line.

Distance from a point P to a plane: $\frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$, where $\mathbf{n}$ is a normal vector of the plane, and Q is a point on the plane.

Line of intersection between two planes: its direction vector is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n_2}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors of the two planes. To find a point on the line, look for (x, y, z) satisfying the equations for both planes.

Angle between two planes: $\cos^{-1}(\frac{\mathbf{n}_1 \cdot \mathbf{n_2}}{\|\mathbf{n}_1\| \cdot \|\mathbf{n_2}\|})$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors of the two planes.

Total time: 80 minutes.
Total points: 100.

Problem 1 ($5 \times 4 = 20$ points). Calculate:

(1): $\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2$, where $\mathbf{u} = \langle -3, -2, 1 \rangle$, $\mathbf{v} = \langle 4, 1, 5 \rangle$

$$(-12 - 2 + 5) + (9 + 4 + 1) = 5$$

(2): $\text{proj}_{\mathbf{v}}\mathbf{u}$, where $\mathbf{u} = \langle -3, -2, 1 \rangle$, $\mathbf{v} = \langle 4, 1, 5 \rangle$

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{-12 - 2 + 5}{16 + 1 + 25}\langle 4, 1, 5 \rangle = \frac{-9}{42}\langle 4, 1, 5 \rangle$$

(3): $\int \langle t^2, \cos(2t), e^{3t} \rangle \, \mathrm{d}t$

$$\langle \frac{1}{3}t^3, \frac{1}{2}\sin(2t), \frac{1}{3}e^{3t} \rangle + \vec{C}$$

(4): $\frac{\partial^2 f}{\partial x^2}$, where $f(x, y) = x^3 \sin(xy)$

$$\frac{\partial f}{\partial x} = 3x^2 \sin(xy) + x^3 \cos(xy)y$$
$$\frac{\partial^2 f}{\partial x^2} = 6x \sin(xy) + 6x^2 \cos(xy)y - x^3 \sin(xy)y^2$$

Problem 2 ($10 + 10 = 20$ points). Calculate:

(1) The distance from $P(1, 0, -2)$ to the line $\mathbf{r}(t) = \langle 2 - t, 3 + 3t, 5t\rangle$.
Direction vector of the line: $\mathbf{v} = \langle -1, 3, 5\rangle$.
A point on the line: $M(2, 3, 0)$.

$$\overrightarrow{PM} = \langle 1, 3, 2\rangle$$

$$\overrightarrow{PM} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 2 \\ -1 & 3 & 5 \end{vmatrix} = \langle 9, -7, 6\rangle$$

$$\text{distance} = \frac{\sqrt{81 + 49 + 36}}{\sqrt{1 + 9 + 25}} = \frac{\sqrt{166}}{\sqrt{35}}$$

(2) The line of intersection of the planes $x - 2y + z = 1$ and $3x - y + 2z = 0$.
Normal vectors of the planes: $\mathbf{n}_1 = \langle 1, -2, 1\rangle$, $\mathbf{n}_2 = \langle 3, -1, 2\rangle$.
Direction vector of line:

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \langle -3, 1, 5\rangle$$

To find a point on the line, set $z = 0$, get $x - 2y = 1$, $3x - y = 0$. Solve to get $x = -\frac{1}{5}$, $y = -\frac{3}{5}$.
Therefore a point is $(-\frac{1}{5}, -\frac{3}{5}, 0)$.
The vector equation of the line is

$$\mathbf{r}(t) = \langle -\frac{1}{5}, -\frac{3}{5}, 0\rangle + t\langle -3, 1, 5\rangle$$

Problem 3 ($8+8+4=20$ points). An object is moving on the xy-plane. At $t=0$, it is located at the origin, and its velocity is $\langle 0,2\rangle$. Its acceleration is given by

$$\mathbf{a}(t)=\langle -4\cos(2t),-4\sin(2t)\rangle$$

(1) Find its velocity function $\mathbf{v}(t)$.

$$\mathbf{v}(t)=\int \mathbf{a}(t)\,\mathrm{d}t=\int \langle -4\cos(2t),-4\sin(2t)\rangle\,\mathrm{d}t=\langle -2\sin(2t),2\cos(2t)\rangle+\vec{C_1}$$

By the initial condition $\mathbf{v}(0)=\langle 0,2\rangle$, we have

$$\langle 0,2\rangle=\langle 0,2\rangle+\vec{C_1}$$

that gives $\vec{C_1}=\vec{0}$. Therefore

$$\mathbf{v}(t)=\langle -2\sin(2t),2\cos(2t)\rangle$$

(2) Find its position function $\mathbf{r}(t)$.

$$\mathbf{r}(t)=\int \mathbf{v}(t)\,\mathrm{d}t=\int \langle -2\sin(2t),2\cos(2t)\rangle\,\mathrm{d}t=\langle \cos(2t),\sin(2t)\rangle+\vec{C_2}$$

By the initial condition $\mathbf{r}(0)=\langle 0,0\rangle$, we have

$$\langle 0,0\rangle=\langle 1,0\rangle+\vec{C_2}$$

that gives $\vec{C_2}=\langle -1,0\rangle$. Therefore

$$\mathbf{r}(t)=\langle -1+\cos(2t),\sin(2t)\rangle$$

(3) Using your result in (2), find the smallest x-coordinate the object can achieve.
The smallest possible value of $\cos(2t)$ is -1. Therefore the smallest possible x-coordinate is $-1-1=-2$.

Problem 4 ($5+5+10=20$ points). Consider the function

$$f(x,y) = y\cos x + 2xy^2$$

(1) Find ∇f.

$$\nabla f = \langle -y\sin x + 2y^2, \cos x + 4xy \rangle$$

(2) Starting from $(\frac{\pi}{2}, 1)$, in which direction does f increase the fastest? Express the direction by a unit vector.
The direction is given by

$$\nabla f(\frac{\pi}{2}, 1) = \langle 1, 2\pi \rangle$$

Normalize:

$$\frac{1}{\sqrt{1+4\pi^2}}\langle 1, 2\pi \rangle$$

(3) Write the linear approximation of f at $(0,2)$, and use it to approximate $f(-0.02, 1.97)$.

$$f_x(0,2) = 8, \quad f_y(0,2) = 1, \quad f(0,2) = 2$$

The linear approximation is

$$L(x,y) = 2 + 8x + (y-2)$$

$$f(-0.02, 1.97) \approx L(-0.02, 1.97) = 2 + 8 \times (-0.02) + (-0.03) = 1.81$$

Problem 5 ($5+10+5=20$ points). Consider the function

$$f(x,y) = x - xy^2 - y^3$$

(1) Find all critical points of f.

$$f_x = 1 - y^2, \quad f_y = -2xy - 3y^2$$

Set both equal to 0. From $1-y^2=0$ we get $y=\pm 1$. If $y=1$, we get $-2x-3=0$, $x=-\frac{3}{2}$; If $y=-1$, we get $2x-3=0$, $x=\frac{3}{2}$. Therefore we get two critical points:

$$(-\frac{3}{2},1), \quad (\frac{3}{2},-1)$$

(2) Determine the type of the critical points (local maximum / local minimum / neither).

$$f_{xx} = 0, \quad f_{xy} = -2y, \quad f_{yy} = -2x - 6y$$

At $(-\frac{3}{2},1)$:

$$f_{xx} = 0, \quad f_{xy} = -2, \quad f_{yy} = -2(-\frac{3}{2}) - 6 = -3$$

$$D = 0 \times (-3) - (-2)^2 = -4 < 0$$

Neither.
At $(\frac{3}{2},-1)$:

$$f_{xx} = 0, \quad f_{xy} = 2, \quad f_{yy} = -2(\frac{3}{2}) + 6 = 3$$

$$D = 0 \times 3 - 2^2 = -4 < 0$$

Neither.

(3) Suppose we want to find the maximum and minimum of f with the constraint $x^4+y^4=1$ by the Lagrange multiplier method. Write the algebraic system of equations one needs to solve for x,y,λ. DO NOT solve it.
Constraint equation

$$g(x,y) = x^4 + y^4 - 1 = 0$$

$$\nabla f = \langle 1-y^2, -2xy-3y^2 \rangle, \quad \nabla g = \langle 4x^3, 4y^3 \rangle$$

Set $\nabla f = \lambda \nabla g$, together with the constraint, we get

$$\begin{cases} 1 - y^2 = \lambda \cdot 4x^3 \\ -2xy - 3y^2 = \lambda \cdot 4y^3 \\ x^4 + y^4 - 1 = 0 \end{cases}$$