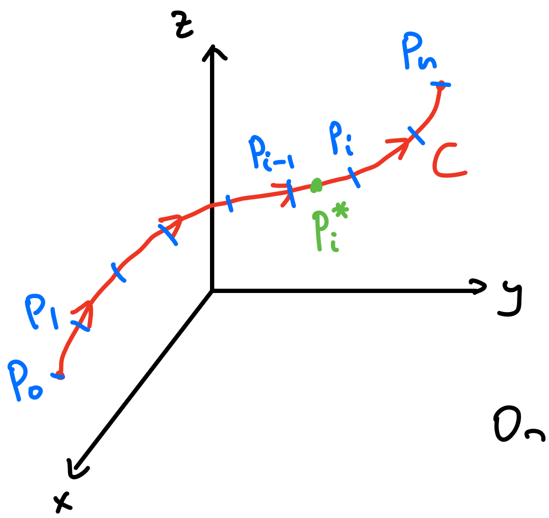
## 6.2 (continued)

- Vector line integral

Motivating example: work done by a force




An object moves along a curve $C$. When it is at $(x,y,z)$, a force $\vec{F}(x,y,z)$ acts on it. Calculate total work done by the force

On the arc $P_{i-1} \rightsquigarrow P_i$ : work is approximately

$$\vec{F}(P_i^*) \cdot \Delta \vec{r}_i \qquad \Delta \vec{r}_i = \overrightarrow{P_{i-1} P_i}$$

$$\Rightarrow \text{ total work } \approx \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \Delta \vec{r}_i$$

take limit $n \to \infty$

**Def** The <u>vector line integral</u> of $\vec{F}$ along $C$ is

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \Delta \vec{r}_i$$

- Here, $C$ is an <u>oriented</u> curve. One needs to specify its direction

- Relation w/ scalar line integral:

$$\Delta \vec{r}_i \approx \vec{T}(P_i^*) \Delta s_i$$

$\vec{T}$ : unit tangent vector

$$\Rightarrow \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \Delta \vec{r}_i \approx \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \vec{T}(P_i^*) \Delta s_i$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$$

- When $C$ is parametrized by $\vec{r}(t)$, $a \le t \le b$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\underbrace{\vec{T}}_{\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\cdot\|\vec{r}'(t)\|\,dt$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt}$$

" $d\vec{r} = \vec{r}'(t)dt$ "

Ex $C$ is the segment from $P(1,1,0)$ to $Q(2,1,3)$.

Calculate $\int_C \langle y, z, -x\rangle\cdot d\vec{r}$

$$\vec{PQ} = \langle 1, 0, 3\rangle$$

$$\vec{r}(t) = \langle 1,1,0\rangle + t\langle 1,0,3\rangle \qquad 0\le t\le 1$$

$$= \langle \underbrace{1+t}_{x}, \underbrace{1}_{y}, \underbrace{3t}_{z}\rangle$$

$$\vec{r}'(t) = \langle 1, 0, 3\rangle$$

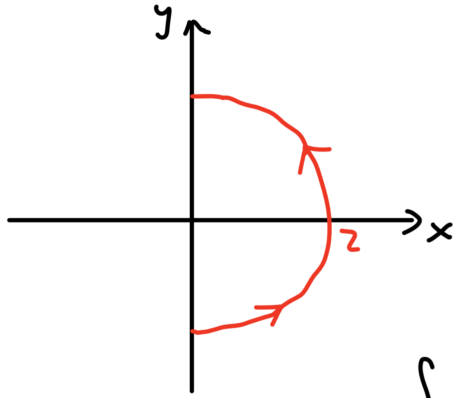
$$\int_C \langle y, z, -x\rangle\cdot d\vec{r} = \int_0^1 \langle 1, 3t, -(1+t)\rangle\cdot\langle 1,0,3\rangle\,dt$$

$$= \int_0^1 \left(1+0-3(1+t)\right)dt = \int_0^1 (-3t-2)\,dt = \left(-\frac{3}{2}t^2-2t\right)\Big|_0^1 = -\frac{7}{2}.$$

- Another notation : when $\vec{F} = \langle P, Q, R\rangle$, one also writes

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

**Ex** Let $C$ be the part of the circle $x^2+y^2=4$ w/ $x \geq 0$, traveling from bottom to top. Calculate $\int_C y\,dx - x\,dy$




$$C: \quad \vec{r}(t) = \langle \underset{x}{2\cos t}, \underset{y}{2\sin t} \rangle \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

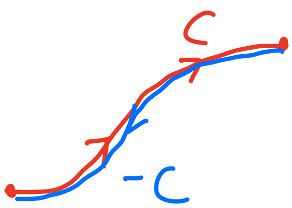
$$\vec{r}\,'(t) = \langle \underset{dx}{-2\sin t}, \underset{dy}{2\cos t} \rangle$$

$$\int_C y\,dx - x\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \big( \underbrace{2\sin t \cdot (-2\sin t) - 2\cos t \cdot 2\cos t}_{-4\sin^2 t - 4\cos^2 t = -4} \big)\,dt$$

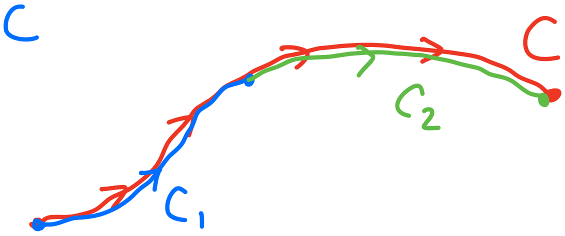
$$= -4\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = -4\pi.$$

- Properties

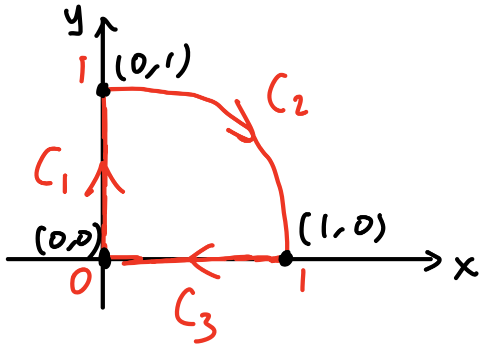
$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$




$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$




**Ex** Let $C$ be the boundary of $\{(x,y) : x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$, traveling clockwise. Calculate $\int_C x\,dx + y\,dy$




$$C_1: \quad \vec{r}(t) = \langle 0,0 \rangle + t\langle 0,1 \rangle \qquad 0 \leq t \leq 1$$

$$= \langle \underset{x}{0}, \underset{y}{t} \rangle$$

$$\vec{r}\,'(t) = \langle \underset{dx}{0}, \underset{dy}{1} \rangle$$

$$\int_{C_1} x\,dx + y\,dy = \int_0^1 (0 \cdot 0 + t \cdot 1)\,dt = \frac{1}{2}t^2 \Big|_0^1 = \frac{1}{2}$$

$$-C_2: \quad \vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle \qquad 0 \le t \le \frac{\pi}{2}$$

$$\vec{r}\,'(t) = \langle \underset{dx}{-\sin t}, \underset{dy}{\cos t} \rangle$$

$$\int_{C_2} x\,dx + y\,dy = -\int_{-C_2} x\,dx + y\,dy = -\int_0^{\frac{\pi}{2}} \left(\cos t \cdot (-\sin t) + \sin t \cdot \cos t\right) dt$$

$$= 0$$

$$C_3: \quad \vec{r}(t) = \langle 1, 0 \rangle + t \langle -1, 0 \rangle \qquad 0 \le t \le 1$$

$$= \langle \underset{x}{1-t}, \underset{y}{0} \rangle$$

$$\vec{r}\,'(t) = \langle \underset{dx}{-1}, \underset{dy}{0} \rangle$$

$$\int_{C_3} x\,dx + y\,dy = \int_0^1 \big(\underbrace{(1-t)\cdot(-1) + 0 \cdot 0}_{-1+t}\big)\, dt = \left(-t + \tfrac{1}{2}t^2\right)\Big|_0^1 = -\frac{1}{2}$$

$$\Rightarrow \int_C x\,dx + y\,dy = \frac{1}{2} + 0 + \left(-\frac{1}{2}\right) = \boxed{0}$$

- Flux of $\vec{F}$ across a 2D curve $C$: defined as

$$\int_C \vec{F} \cdot \vec{N}\, ds$$

where $\vec{N}$ is the unit normal vector of $C$

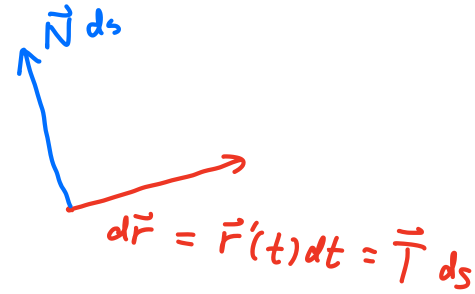
↳ needs to specify "up / down"


When $\vec{F}$ is a fluid velocity field, the flux gives the amount of fluid passing $C$ per unit time

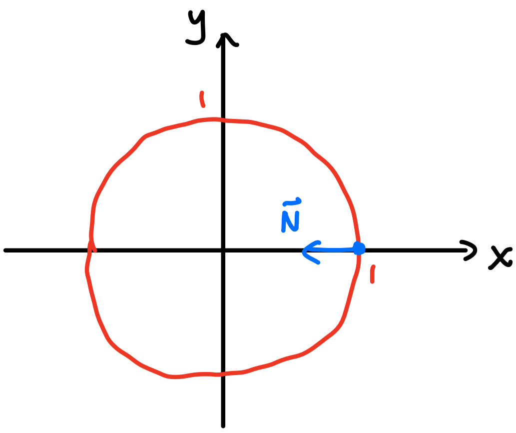
- To calculate flux, if $\vec{r}(t) = \langle x(t), y(t) \rangle$ then

$$\vec{N}\,ds = \pm \langle y'(t), -x'(t) \rangle\, dt$$

↳ match direction of $\vec{N}$




<u>Ex</u> Let $C$ be unit circle, w/ inward $\vec{N}$. Calculate flux of $\vec{F}(x,y) = \langle -x, -y \rangle$ across $C$.




$C: \quad \vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle \quad 0 \le t \le 2\pi$

$\vec{r}'(t) = \langle \underset{x'}{-\sin t}, \underset{y'}{\cos t} \rangle$

$\vec{N}\,ds = -\langle \cos t, \sin t \rangle\, dt$

$$\int_C \vec{F} \cdot \vec{N}\, ds = \int_0^{2\pi} \langle -\cos t, -\sin t \rangle \cdot \left( -\langle \cos t, \sin t \rangle \right) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = 2\pi.$$