6.3 Conservative vector fields (also see sec. 6.1)

Def A vector field $\vec{F}$ is a conservative field (gradient field) if there exists a scalar function f such that

$$\vec{F} = \nabla f$$

In this case, f is called a potential function of $\vec{F}$

↳ "similar to anti-derivative"

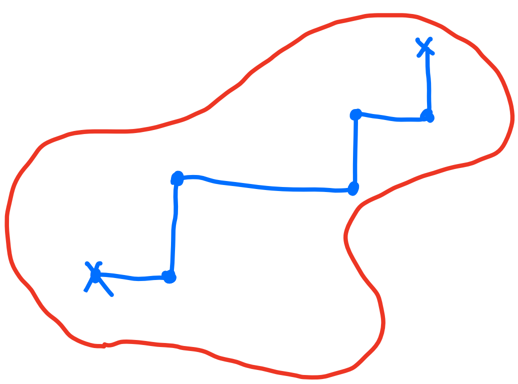
For example, $\vec{F}(x,y) = \langle x, y \rangle$ is conservative.

A potential function is $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$\vec{G}(x,y) = \langle y, x \rangle$ is conservative.

A potential function is $g(x,y) = xy$.

Thm Let $\vec{F}$ be defined on an open, connected region. If $\vec{F}$ is conservative, and f_1, f_2 are potential functions of $\vec{F}$, then $f_1 = f_2 + C$ for some constant C.



Proof $\vec{F} = \nabla f_1 = \nabla f_2$

$$\nabla(\underbrace{f_1 - f_2}_{g}) = 0$$

Say, in 2D, $g_x = g_y = 0$

$\Rightarrow$ g is constant on horizontal / vertical segments in domain

$\Rightarrow$ $g = C$.

__Thm__ (cross derivative property)

2D version: If $\vec{F}(x,y) = \langle P, Q \rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

__Proof__ If $\vec{F}$ is conservative, then $\vec{F} = \nabla f$

$$\Rightarrow P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

equal.

3D version: If $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ is conservative, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

__Ex__ Show that $\vec{F} = \langle \underbrace{x \sin y}_{P}, \underbrace{\frac{x^2}{2} \cos y + z}_{Q}, \underbrace{z^2 y}_{R} \rangle$ is not conservative

$$\frac{\partial P}{\partial y} = x \cos y \qquad \frac{\partial Q}{\partial x} = x \cos y \qquad \checkmark$$

$$\frac{\partial P}{\partial z} = 0 \qquad \frac{\partial R}{\partial x} = 0 \qquad \checkmark$$

$$\frac{\partial Q}{\partial z} = 1 \qquad \frac{\partial R}{\partial y} = z^2 \qquad \times \qquad \Rightarrow \vec{F} \text{ is not conservative.}$$

• How to find a potential function for a conservative field ?

Ex Given that $\vec{F}(x,y) = \langle \underset{\swarrow f_x}{e^{-x}y^2}, \underset{\swarrow f_y}{-2e^{-x}y + y} \rangle$ is conservative, find a potential function

① Integrate f_x in x (viewing y as constant)

$$f(x,y) = \int e^{-x}y^2\,dx = -e^{-x}y^2 + \underbrace{g(y)}$$

↳ a "constant" depending on y

② Take y- derivative

$$-2e^{-x}y + y = f_y = -e^{-x}\cdot 2y + g'(y)$$

$$g'(y) = y \qquad g(y) = \int y\,dy = \tfrac{1}{2}y^2 + C$$

$$\Rightarrow \quad f(x,y) = -e^{-x}y^2 + \tfrac{1}{2}y^2 + C$$

Ex Given that $\vec{F}(x,y,z) = \langle \underset{\swarrow f_x}{2xyz - z}, \underset{\swarrow f_y}{x^2z - y}, \underset{\swarrow f_z}{x^2y - x} + \sin z \rangle$ is conservative, find a potential function.

① Integrate f_x in x (viewing y, z as constant)

$$f(x,y,z) = \int (2xyz - z)\,dx = x^2yz - xz + g(y,z)$$

② Take y- derivative

$$x^2z - y = f_y = x^2z + g_y$$

$$g_y = -y$$

Integrate in y (viewing z as constant)

$$g(y,z) = \int -y\,dy = -\tfrac{1}{2}y^2 + h(z)$$

$$\Rightarrow \quad f(x,y,z) = x^2yz - xz - \frac{1}{2}y^2 + h(z)$$

③ Take z-derivative

$$x^2y - x + \sin z = f_z = x^2y - x + h'(z)$$

$$h'(z) = \sin z \qquad h(z) = \int \sin z \, dz = -\cos z + C$$

$$\Rightarrow \quad f(x,y,z) = x^2yz - xz - \frac{1}{2}y^2 - \cos z + C$$

- Line integral of conservative fields

Thm Let C be is a curve from A to B, and $\vec{F} = \nabla f$.

Then
$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Proof C : $\vec{r}(t)$, $a \le t \le b$

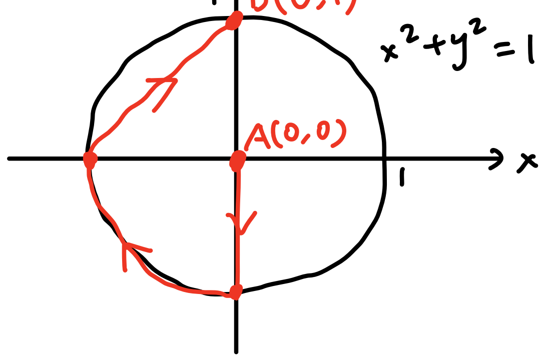
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$

Ex Given $\vec{F}(x,y) = \langle e^{-x}y^2, -2e^{-x}y + y \rangle = \nabla f$,

$f(x,y) = -e^{-x}y^2 + \frac{1}{2}y^2$. Let the curve C be the following:



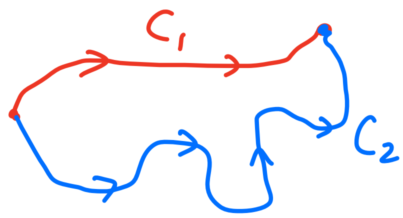
Calculate $\int_C \vec{F} \cdot d\vec{r}$

$$= f(B) - f(A)$$

$$= \left(-e^{-0} \cdot 1^2 + \frac{1}{2} \cdot 1^2\right) - 0 = -\frac{1}{2}.$$

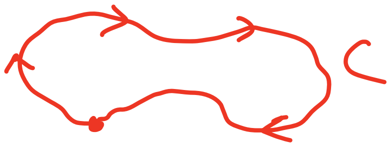
- Consequence of Theorem : If $\vec{F}$ is conservative,

① $\int_C \vec{F}\cdot d\vec{r}$ only depends on endpoints of C, but not the path.



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

② If C is a closed curve, then $\int_C \vec{F}\cdot d\vec{r} = 0$



- conservative ⟺ ① ⟺ ②

- If $\vec{F}$ is defined on a <u>simply-connected</u> domain, then

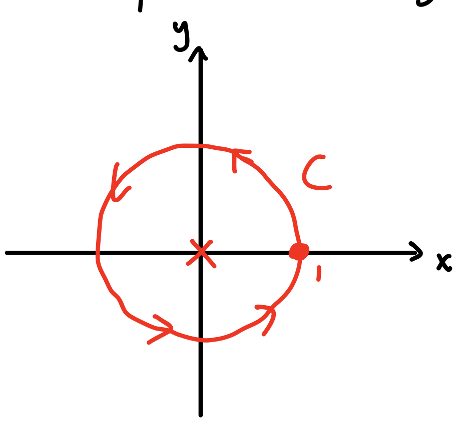
"cross derivative property" ⟺ conservative

Any two curves from A to B can be continuously transformed from one to another

- The above is NOT true for general domain.

Example: $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ defined on $\{(x,y) : (x,y) \neq (0,0)\}$

P Q



$$\frac{\partial P}{\partial y} = \frac{-1\cdot(x^2+y^2) - (-y)\cdot 2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)}$$

⇒ Cross derivative property holds.

$C: \quad \vec{r}(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t} \rangle \qquad 0 \le t \le 2\pi$

$\qquad \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi \neq 0 \quad \Rightarrow \text{NOT conservative.}$$