6.4 Green's Theorem

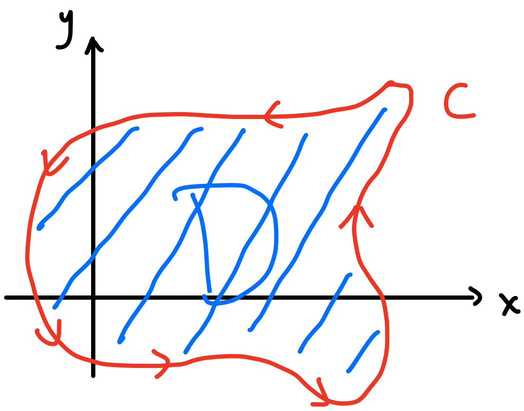
Recall: in 2D

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{"}\Leftrightarrow\text{"} \quad \vec{F} = \langle P, Q \rangle \text{ conservative} \quad \Leftrightarrow \quad \int_C \vec{F} \cdot d\vec{r} = 0$$

for closed curves C

doesn't intersect itself

Thm Let C be a simple closed curve in the plane, traveling ccw, enclosing the region D. Let $\vec{F}(x,y) = \langle P, Q \rangle$.



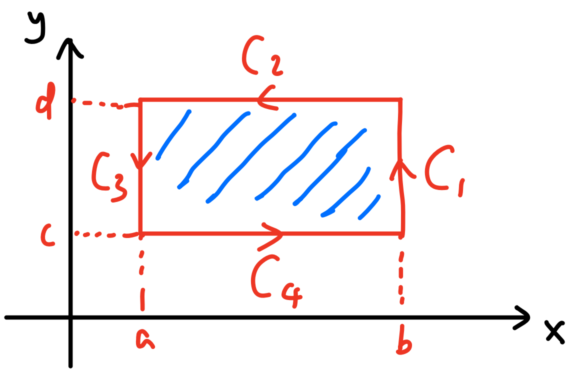
Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

put "o" to indicate that the curve is closed.

Compare: $G(b) - G(a) = \int_a^b g(x)\, dx$

Proof (for rectangular region)



C_1: $\vec{r}(t) = \langle b, t \rangle, \quad c \le t \le d$

$\vec{r}\,'(t) = \langle 0, 1 \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_c^d \langle P(b,t), Q(b,t) \rangle \cdot \langle 0, 1 \rangle\, dt$$

$$= \int_c^d Q(b,t)\, dt$$

$-C_3$: $\vec{r}(t) = \langle a, t \rangle, \quad c \le t \le d$

$\vec{r}\,'(t) = \langle 0, 1 \rangle$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = -\int_{-C_3} \vec{F} \cdot d\vec{r} = -\int_c^d \langle P(a,t), Q(a,t) \rangle \cdot \langle 0, 1 \rangle\, dt$$

$$= -\int_c^d Q(a,t)\, dt$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} = \int_c^d Q(b,t)\,dt - \int_c^d Q(a,t)\,dt$$

$$= \int_c^d \big(Q(b,t) - Q(a,t)\big)\,dt$$

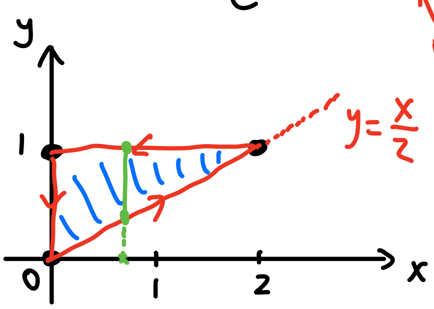
$$= \int_c^d \int_a^b \frac{\partial Q}{\partial x}(x,y)\,dx\,dy$$

$$= \iint_D \frac{\partial Q}{\partial x}\,dA$$

Similarly, $\displaystyle\int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} = \iint_D \left(-\frac{\partial P}{\partial y}\right) dA$

<u>Ex</u> Let C be the boundary of the triangle w/ vertices $(0,0)$, $(0,1)$, $(2,1)$, traveling ccw. Calculate

$$\oint_C \langle \underbrace{e^{x^3}+xy}_{P},\ \underbrace{x-\sin(e^{-y})}_{Q} \rangle \cdot d\vec{r}$$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - x$$

Green's $\quad = \iint_D (1-x)\,dA$

$$= \int_0^2 \int_{\frac{x}{2}}^1 (1-x)\,dy\,dx$$

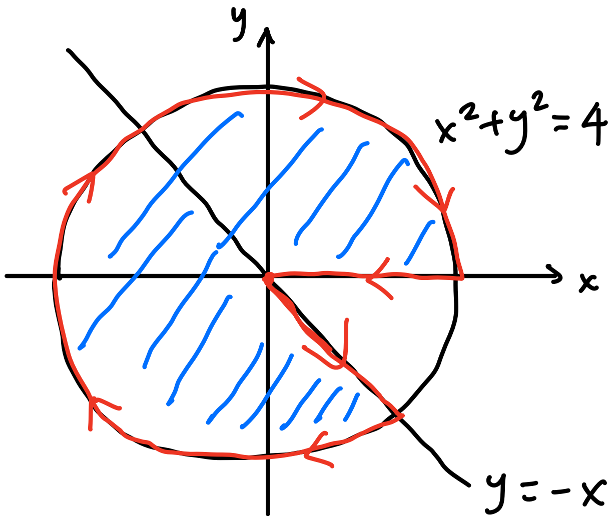
$$= \int_0^2 (1-x)\left(1-\frac{x}{2}\right) dx$$

$$= \int_0^2 \left(1 - \frac{3}{2}x + \frac{1}{2}x^2\right) dx$$

$$= \left(x - \frac{3}{4}x^2 + \frac{1}{6}x^3\right)\Big|_0^2$$

$$= 2 - \frac{3}{4}\cdot 4 + \frac{1}{6}\cdot 8 = \frac{1}{3}.$$

Ex Let the curve C be the following:



Calculate

$$\oint_C (\underbrace{\cos x \sin y + y^2}_{P})\,dx + (\underbrace{\sin x \cos y - y^4}_{Q})\,dy$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \cos x \cos y - (\cos x \cos y + 2y)$$

$$= -2y$$

Green's

$$= -\iint_D (-2y)\,dA = \iint_D 2y\,dA$$

C is cw direction!

$$= \int_0^2 \int_0^{\frac{7\pi}{4}} 2 r \sin\theta \; r\,d\theta\,dr$$

$$= 2 \underbrace{\int_0^2 r^2\,dr}_{\frac{1}{3}r^3\Big|_0^2 = \frac{8}{3}} \cdot \underbrace{\int_0^{\frac{7\pi}{4}} \sin\theta\,d\theta}_{-\cos\theta\Big|_0^{\frac{7\pi}{4}} = -\frac{\sqrt{2}}{2} - (-1)} = \boxed{\frac{16}{3}\left(1 - \frac{\sqrt{2}}{2}\right)}$$

6.5 Divergence and curl

Def Let $\vec{F}$ be a vector field. The divergence of $\vec{F}$ is

$$\nabla \cdot \vec{F} \quad \text{(also denoted as div } \vec{F}\text{)}$$

Recall: in 2D

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\text{"} \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \text{"}$$

in 3D

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\text{"} \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \text{"}$$

in 2D, $\vec{F} = \langle P, Q \rangle$, $\nabla \cdot \vec{F} = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$

in 3D, $\vec{F} = \langle P, Q, R \rangle$

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

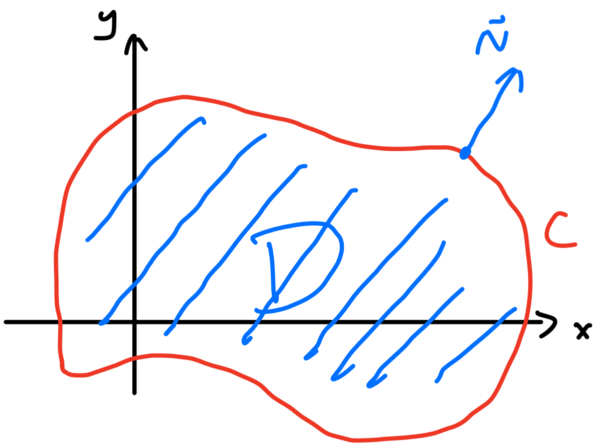
Ex $\vec{F}(x,y) = \langle x^2 y^3, \sin x \sin(2y) \rangle$

$$\nabla \cdot \vec{F} = 2xy^3 + \sin x \cos(2y) \cdot 2$$

$$\operatorname{div}(\langle y, z, -x \rangle) = 0 + 0 + 0 = 0$$

Thm (Flux form of Green's Thm)

Let C be a simple closed curve in the plane, w/ outward normal, enclosing the region D. Let $\vec{F} = \langle P, Q \rangle$



Then $$\oint_C \vec{F} \cdot \vec{N} \, ds = \iint_D \nabla \cdot \vec{F} \, dA$$