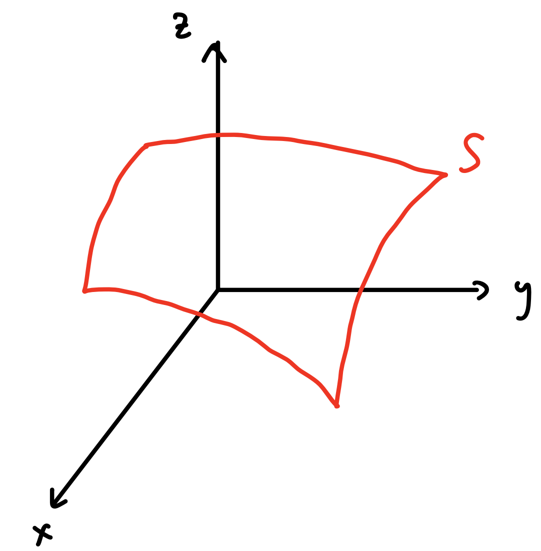
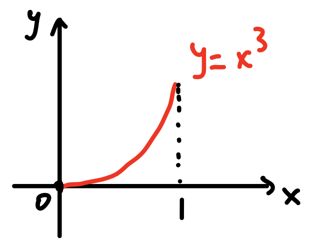
6.6 Surface integrals



Quiz 8 # 1



$\vec{r}(t) = \langle t, t^3 \rangle$

$0 \le t \le 1$

Goal: define integrals over a surface S

Applications:

surface area

mass of an object that occupies S

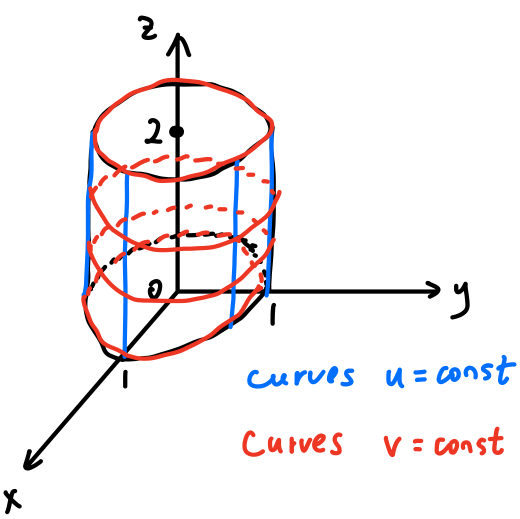
flux across S

- Parametrization of S: we need two parameters.

$$S: \quad \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \qquad (u,v) \in D$$

parameter domain
a subset of $\mathbb{R}^2$

<u>Ex</u> Let S be the side surface of the cylinder $x^2 + y^2 = 1$, between $z = 0$ and $z = 2$. Parametrize S.



Cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad (r, \theta, z)$$

On S, r is constant 1 $\Rightarrow$

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases}$$

rename $\theta \rightsquigarrow u$, $z \longrightarrow v$,

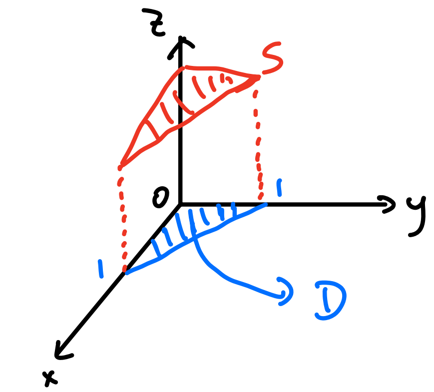
$\vec{r}(u, v) = \langle \cos u, \sin u, v \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 2$

- Parametrize a graph : use (x, y) as parameters

<u>Ex</u> Let S be the part of the plane $3x + y + 2z = 6$ with $x \ge 0$, $y \ge 0$, $x + y \le 1$. Parametrize S.

$2z = 6 - 3x - y$

$z = \frac{1}{2}(6 - 3x - y)$



$\vec{r}(u, v) = \langle u, v, \frac{1}{2}(6 - 3u - v) \rangle$

$(u, v) \in D \qquad D = \{(u, v) : u \ge 0, v \ge 0, u + v \le 1\}$.

<u>Ex</u> Parametrize the upper half of unit sphere

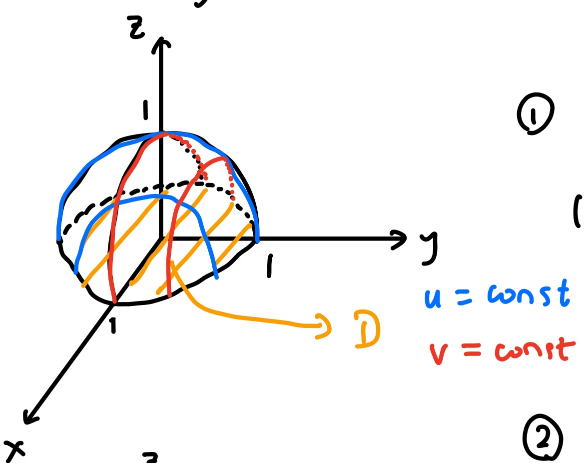
$x^2 + y^2 + z^2 = 1$

$z^2 = 1 - x^2 - y^2$

$z = \sqrt{1 - x^2 - y^2}$

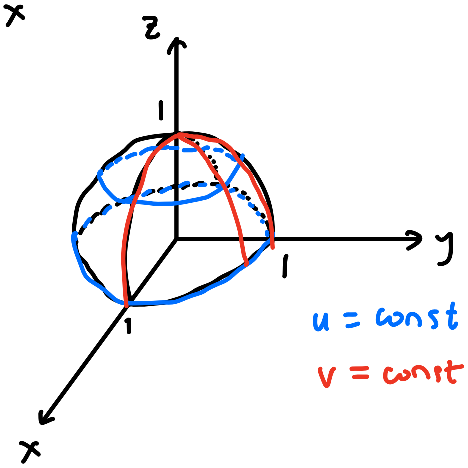
① as a graph

② by spherical coordinates



① $\vec{r}(u, v) = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$

$(u, v) \in D, \quad D = \{(u, v) : u^2 + v^2 \le 1\}$



② Spherical coordinates :

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

On S, ρ is constant 1.

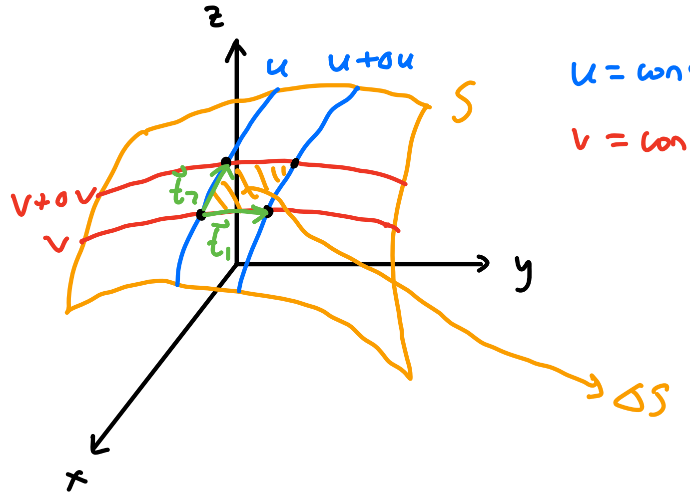
$$\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases}$$

rename $\varphi \rightsquigarrow u$, $\theta \rightsquigarrow v$

$$\vec{r}(u,v) = \langle \sin u\cos v,\ \sin u\sin v,\ \cos u \rangle$$

$$0 \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi$$

- Surface area element



u = const

v = const

$$\vec{t}_1 = \vec{r}(u+\Delta u, v) - \vec{r}(u,v)$$
$$\approx \vec{r}_u(u,v)\,\Delta u$$
$$\vec{t}_2 = \vec{r}(u, v+\Delta v) - \vec{r}(u,v)$$
$$\approx \vec{r}_v(u,v)\,\Delta v$$

$$\Rightarrow \Delta S \approx \|\vec{t}_1 \times \vec{t}_2\| \approx \|\vec{r}_u \times \vec{r}_v\|\,\Delta u\,\Delta v$$

area formula for parallelogram

Compare : on a curve $\vec{r}(t)$

" $ds = \|\vec{r}\,'(t)\|\,dt$ "

$\Rightarrow$ " $dS = \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$ "

- Surface area : If S is parametrized by $\vec{r}(u,v)$, $(u,v) \in D$ then its surface area is

$$\iint_D \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$$

<u>Ex</u> Let S be the side surface of the cylinder $x^2+y^2=1$, between $z=0$ and $z=2$. Find surface area of S

Previously :

$$\vec{r}(u,v) = \langle \cos u, \sin u, v \rangle \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 2$$

$$\vec{r}_u = \langle -\sin u, \cos u, 0 \rangle$$

$$\vec{r}_v = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u, \sin u, 0 \rangle$$

$$\| \vec{r}_u \times \vec{r}_v \| = \sqrt{\cos^2 u + \sin^2 u + 0} = 1$$

$$\Rightarrow \text{Surface area} = \iint_D 1 \, dA = \int_0^{2\pi} \int_0^{2} 1 \, dv\,du$$

$$= \int_0^{2\pi} 1 \, du \cdot \int_0^{2} 1 \, dv = 2\pi \cdot 2 = 4\pi$$