6.6 (continued)

Recall:

- Surface area: If S is parametrized by $\vec{r}(u,v)$, $(u,v) \in D$ then its surface area is

$$\iint_D \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

Ex Area of the upper half of unit sphere

Previously:

$$\vec{r}(\overset{\varphi}{u}, \overset{\theta}{v}) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \qquad \underbrace{0 \le u \le \frac{\pi}{2}, \; 0 \le v \le 2\pi}_{D}$$

$$\vec{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$$

$$\vec{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix}$$

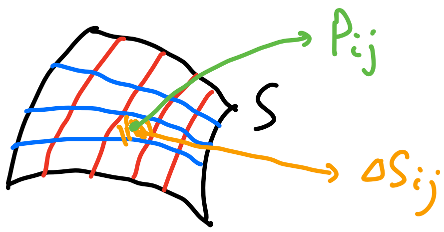
$$= \langle \sin^2 u \cos v, \sin^2 u \sin v, \underbrace{\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v}_{\cos u \sin u} \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\underbrace{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v}_{\sin^4 u} + \cos^2 u \sin^2 u}$$

$$= \sqrt{\sin^2 u (\sin^2 u + \cos^2 u)} = \sin u$$

$$\text{Surface area} = \iint_D \sin u \, dA = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin u \; dv\, du = \underbrace{\int_0^{\frac{\pi}{2}} \sin u \, du}_{-\cos u \Big|_0^{\frac{\pi}{2}} = 1} \cdot \underbrace{\int_0^{2\pi} 1 \, dv}_{2\pi}$$

$$= \boxed{2\pi}$$



Def The surface integral of a scalar function f over a surface S is

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}$$

- For a sheet occupying S w/ density function f, its mass is given by $\iint_S f(x,y,z)\,dS$
- Calculation by parametrization : $\vec{r}(u,v)$, $(u,v)\in D$

use "$dS = \|\vec{r}_u \times \vec{r}_v\|\,du\,dv$"

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,\|\vec{r}_u\times\vec{r}_v\|\,dA$$

Ex Let S be the upper half of unit sphere. Calculate $\iint_S z\,dS$

Previously :

$$\vec{r}(\underset{\varphi}{u},\underset{\theta}{v}) = \langle \underbrace{\sin u\cos v}_{x},\ \underbrace{\sin u \sin v}_{y},\ \underbrace{\cos u}_{z}\rangle \qquad \underbrace{0\le u\le \tfrac{\pi}{2},\ 0\le v\le 2\pi}_{D}$$

$$\|\vec{r}_u\times\vec{r}_v\| = \sin u$$

$$\iint_S z\,dS = \iint_D \cos u\cdot\sin u\,dA = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \cos u \sin u\,dv\,du$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \cos u \sin u\,du}_{\frac{1}{2}\sin^2 u\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}} \cdot \underbrace{\int_0^{2\pi} 1\,dv}_{2\pi} = \boxed{\pi}$$

$$\int \cos u \sin u \, du = \int w \, dw = \frac{1}{2} w^2 + C$$

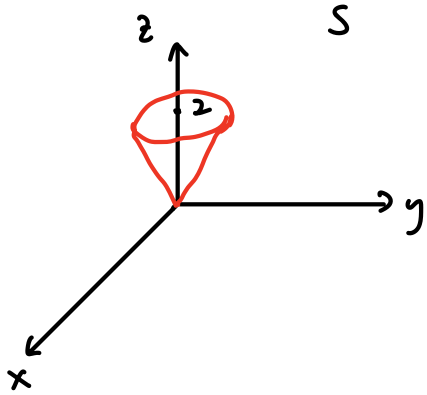
$w = \sin u$

$dw = \cos u \, du$

$$= \frac{1}{2} \sin^2 u + C$$

Ex Let S be the part of the cone $\underbrace{z = 2\sqrt{x^2+y^2}}_{z = 2r}$ below $z = 2$.

Calculate $\iint_S x^2 \, dS$



① Cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

On S, we have $z = 2r$

$$\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 2r \end{cases}$$

rename $r \rightsquigarrow u$, $\theta \longrightarrow v$

$$\vec{r}(u,v) = \langle \underset{x}{u\cos v}, \underset{y}{u\sin v}, \underset{z}{2u} \rangle \qquad \underbrace{0 \le u \le 1, \quad 0 \le v \le 2\pi}_{D}$$

② $\vec{r}_u = \langle \cos v, \sin v, 2 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \langle -2u\cos v, -2u\sin v, \underbrace{u\cos^2 v + u\sin^2 v}_{u} \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\underbrace{4u^2\cos^2 v + 4u^2\sin^2 v}_{4u^2} + u^2} = \sqrt{5}\, u$$

(3) $$\iint_S x^2\,dS = \iint_D u^2\cos^2 v\cdot\sqrt{5}\,u\,dA = \int_0^1\int_0^{2\pi}\sqrt{5}\,u^3\cos^2 v\,dv\,du$$

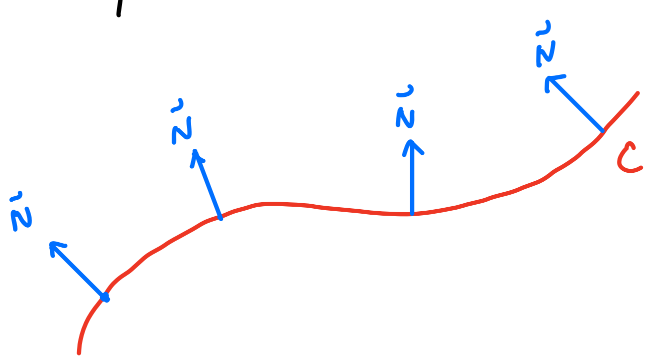
$$= \sqrt{5}\ \underbrace{\int_0^1 u^3\,du}_{\frac{1}{4}u^4\big|_0^1=\frac{1}{4}}\cdot\underbrace{\int_0^{2\pi}\cos^2 v\,dv}_{\int_0^{2\pi}\frac{1}{2}(1+\cos(2v))dv} = \boxed{\frac{\sqrt{5}}{4}\pi}$$

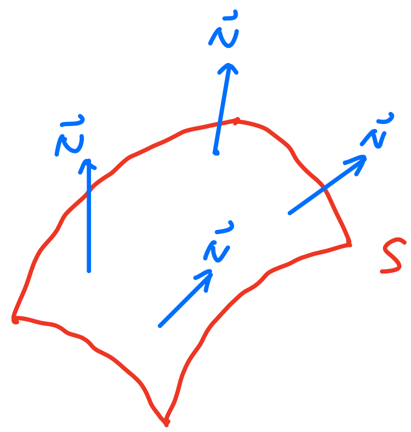
$$= \frac{1}{2}\left(v+\frac{1}{2}\sin(2v)\right)\Big|_0^{2\pi} = \pi$$

Flux integral over a surface "$\iint_S \vec{F}\cdot\vec{N}\,dS$"

- One needs an <u>orientation</u> of the surface S, that is, a consistent choice of unit normal $\vec{N}$.

Compare: 2D | 3D





- Choosing an orientation is possible for some surfaces (called <u>orientable</u> surfaces). Such a surface w/ a given orientation is called an <u>oriented</u> surface

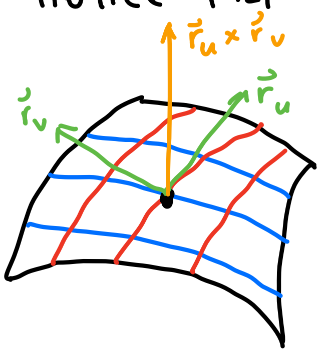
- There are non-orientable surfaces, for example, Mobius band.

<u>Def</u> Let $\vec{F}$ be a vector field and S be an oriented surface. Then the <u>surface integral (flux integral)</u> of $\vec{F}$ over S is

unit normal given by orientation $$\iint_S \vec{F}\cdot\vec{N}\,dS \qquad \left(\text{also denoted } \iint_S \vec{F}\cdot d\vec{S}\right)$$

• To calculate by parametrization $\vec{r}(u,v)$, $(u,v)\in D$

notice that $\vec{r}_u \times \vec{r}_v$ is normal to surface



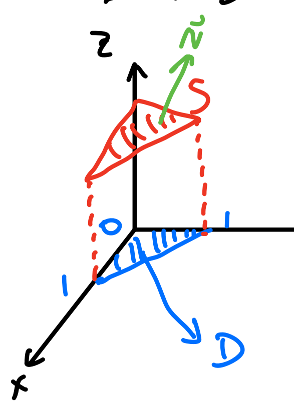
$$\Rightarrow \vec{N}\,dS = \pm\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}\|\vec{r}_u\times\vec{r}_v\|\,du\,dv = \pm(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

determined by orientation

$$\vec{N}\,dS = \pm(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \pm\iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

Ex Let S be the part of the plane $3x+y+2z=6$ w/ $x\ge 0$, $y\ge 0$, $x+y\le 1$, w/ upward normal



Calculate $\iint_S \langle y, 0, 3z\rangle\cdot\vec{N}\,dS$

Previously:

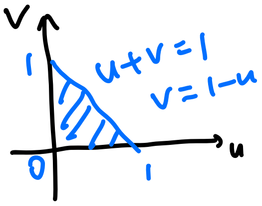
$$\vec{r}(u,v) = \langle \overset{x}{u}, \overset{y}{v}, \overset{z}{\tfrac{1}{2}(6-3u-v)}\rangle,\quad (u,v)\in D$$

$$\vec{r}_u = \langle 1, 0, -\tfrac{3}{2}\rangle \qquad \vec{r}_v = \langle 0, 1, -\tfrac{1}{2}\rangle$$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2}\end{vmatrix} = \langle \tfrac{3}{2}, \tfrac{1}{2}, 1\rangle$$

→ upward (same as $\vec{N}$)

$$\iint_S \langle y,0,3z\rangle\cdot\vec{N}\,dS = \iint_D \langle v, 0, 3\cdot\tfrac{1}{2}(6-3u-v)\rangle\cdot\langle\tfrac{3}{2},\tfrac{1}{2},1\rangle\,dA$$



$$= \iint_D \Big(\underbrace{\tfrac{3}{2}v + 0 + \tfrac{3}{2}(6-3u-v)}_{9-\frac{9}{2}u} \Big)\, dA$$

$$= \int_0^1 \int_0^{1-u} \left(9 - \frac{9}{2}u\right) dv\, du$$

$$= \int_0^1 \left(9 - \frac{9}{2}u\right)(1-u)\, du$$

$$= \frac{9}{2}\int_0^1 \underbrace{(2-u)(1-u)}_{2-3u+u^2}\, du$$

$$= \frac{9}{2}\left(2u - \frac{3}{2}u^2 + \frac{1}{3}u^3\right)\Big|_0^1 = \frac{9}{2}\left(2 - \frac{3}{2} + \frac{1}{3}\right) = \boxed{\frac{15}{4}}$$