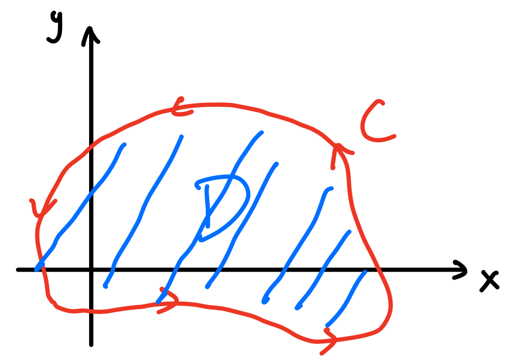
6.7 Stokes' Theorem

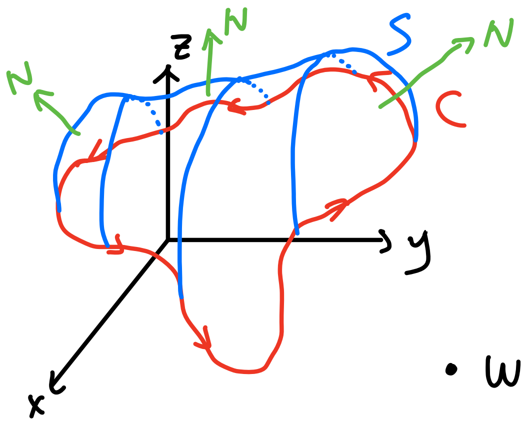
Recall : Green's Theorem (circulation form)



$$\vec{F} = \langle P, Q \rangle$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Stokes' Theorem



Let S be an oriented surface, whose boundary is a simple closed curve C w/ positive orientation. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{N} \, dS$$

- When S is inside xy-plane, Stokes' reduces to Green's.

$\vec{N} = \vec{k}$ if C is ccw

$\vec{F} = \langle P, Q, R \rangle$, $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

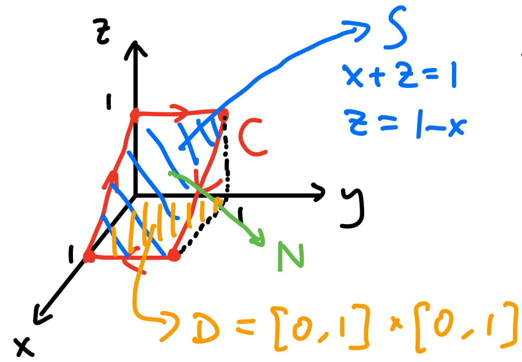
$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} , \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} , \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$(\nabla \times \vec{F}) \cdot \vec{N} = (\nabla \times \vec{F}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Ex Let C be the closed curve consisting of segments, connecting $(1,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,0)$ in order.

Calculate $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = \langle \sin(e^x) + y , \cos(e^{-y}) - z , 2y + z^4 \rangle$$



$$\nabla\times\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(e^x)+y, & \cos(e^{-y})-z, & 2y+z^4 \end{vmatrix}$$

$$= \langle 2-(-1), -(0-0), 0-1 \rangle$$

$$= \langle 3, 0, -1 \rangle$$

$$\oint_C \vec F\cdot d\vec r = \iint_S (\nabla\times\vec F)\cdot\vec N\, dS = \iint_S \langle 3,0,-1\rangle\cdot\vec N\, dS$$

$S: \quad \vec r(u,v) = \langle u, v, 1-u\rangle \qquad (u,v)\in D$

$$\vec r_u = \langle 1, 0, -1\rangle$$

$$\vec r_v = \langle 0, 1, 0\rangle$$

$$\vec r_u\times\vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1, 0, 1\rangle$$

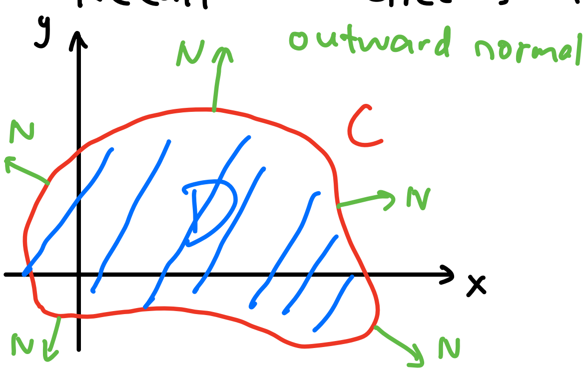
↳ opposite as $\vec N$

$$\iint_S \langle 3,0,-1\rangle\cdot\vec N\, dS = -\iint_D \underbrace{\langle 3,0,-1\rangle\cdot\langle 1,0,1\rangle}_{3+0+(-1)=2} dA$$

$$= -2\,\text{Area}(D) = -2$$

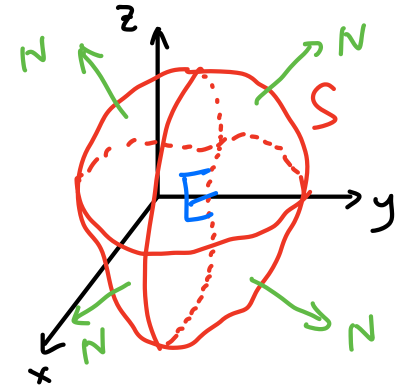
6.8 Divergence Theorem

Recall : Green's Theorem (flux form)



$$\oint_C \vec F\cdot\vec N\, ds = \iint_D (\nabla\cdot\vec F)\, dA$$

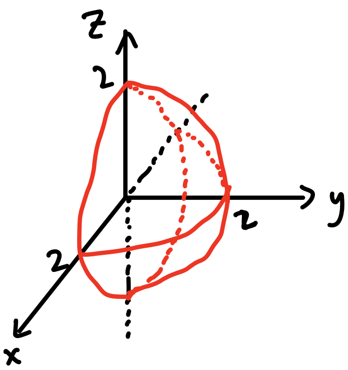
Divergence Theorem



Let S be a closed surface, enclosing the region E, w/ outward normal. Then

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_E (\nabla\cdot\vec{F})\,dV$$

Ex Let $E = \{(x,y,z) : x^2+y^2+z^2 \le 4,\ y \ge 0\}$ and S be its boundary w/ inward normal. Calculate $\iint_S \vec{F}\cdot\vec{N}\,dS$ where $\vec{F} = \langle x^3+yz,\ y^3+zx,\ z^3+xy \rangle$



$0 \le \rho \le 2$

$0 \le \varphi \le \pi$

$0 \le \theta \le \pi$

$$\nabla\cdot\vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iint_S \vec{F}\cdot\vec{N}\,dS = -\iiint_E (3x^2+3y^2+3z^2)\,dV$$

$$= -\int_0^2\int_0^\pi\int_0^\pi 3\rho^2 \cdot \rho^2\sin\varphi\,d\theta\,d\varphi\,d\rho$$

$$= -3\underbrace{\int_0^2 \rho^4\,d\rho}_{\frac{\rho^5}{5}\Big|_0^2 = \frac{32}{5}} \cdot \underbrace{\int_0^\pi \sin\varphi\,d\varphi}_{-\cos\varphi\Big|_0^\pi = 1-(-1)=2} \cdot \underbrace{\int_0^\pi 1\,d\theta}_{\pi} = \boxed{-\frac{192}{5}\pi}$$

Review

Ex S is the cone $z=\sqrt{x^2+y^2}$ w/ $0 \le x \le 2$, $0 \le y \le 1$.

① Parametrize S

② Calculate surface area of S $\quad \iint_S 1 \, dS = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA$

$$S: \quad \vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2} \rangle \qquad \underbrace{0 \le u \le 2, \ 0 \le v \le 1}_{D}$$

$$\vec{r}_u = \langle 1, 0, \tfrac{1}{2}(u^2+v^2)^{-\frac{1}{2}} \cdot 2u \rangle = \langle 1, 0, \frac{u}{\sqrt{u^2+v^2}} \rangle$$

$$\vec{r}_v = \langle 0, 1, \tfrac{1}{2}(u^2+v^2)^{-\frac{1}{2}} \cdot 2v \rangle = \langle 0, 1, \frac{v}{\sqrt{u^2+v^2}} \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = \langle -\frac{u}{\sqrt{u^2+v^2}}, -\frac{v}{\sqrt{u^2+v^2}}, 1 \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{2}$$

$$\text{surface area} = \iint_D \sqrt{2} \, dA = \sqrt{2} \, \text{Area}(D) = 2\sqrt{2}$$