Review

Ex Show that $\vec{F}(x,y) = \langle \underbrace{\tfrac{3}{2}x^2 \sin(2y)}_{P = f_x},\ \underbrace{(x^3-1)\cos(2y)}_{Q = f_y} \rangle$

is conservative by cross derivative property.

Find a potential function.

$$\underbrace{\frac{\partial Q}{\partial x} = 3x^2\cos(2y) \qquad \frac{\partial P}{\partial y} = \frac{3}{2}x^2 \cdot \cos(2y)\cdot 2}_{\text{equal.}}$$

To find a potential function $f(x,y)$:

① Integrate f_x in x, viewing y as constant

$$f(x,y) = \int \frac{3}{2}x^2 \sin(2y)\,dx = \underbrace{\frac{3}{2}\sin(2y)\cdot \frac{1}{3}x^3}_{\frac{1}{2}\sin(2y)\,x^3} + g(y)$$

② Take $\frac{\partial}{\partial y}$

$$(x^3-1)\cos(2y) = f_y = \frac{1}{2}\cos(2y)\cdot 2x^3 + g'(y)$$

$$g'(y) = -\cos(2y)$$

$$g(y) = \int -\cos(2y)\,dy = -\frac{1}{2}\sin(2y) + C$$

$$\Rightarrow f(x,y) = \frac{1}{2}\sin(2y)x^3 - \frac{1}{2}\sin(2y) + C.$$

Ex C: segment from $\underset{A}{(1,0)}$ to $\underset{B}{(0,2)}$. Calculate

$$\int_C xy\,dx + y\,dy \qquad\qquad \vec{AB} = \langle -1, 2\rangle$$

$$C: \quad \vec{r}(t) = \langle 1,0\rangle + t\langle -1,2\rangle \qquad 0 \le t \le 1$$

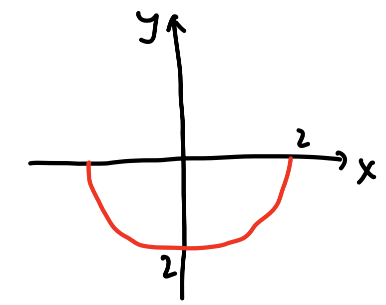
$$= \langle \overset{x}{1-t}, \overset{y}{2t}\rangle$$

$$\vec{r}\,'(t) = \langle \overset{dx}{-1}, \overset{dy}{2}\rangle$$

$$\int_C xy\,dx + y\,dy = \int_0^1 \Big(\underbrace{(1-t)\cdot 2t\cdot(-1) + 2t\cdot 2}_{-2t+2t^2+4t \,=\, 2t^2+2t}\Big)\,dt$$

$$= \left(\tfrac{2}{3}t^3 + t^2\right)\Big|_0^1 = \frac{2}{3} + 1 = \frac{5}{3}.$$

Ex C: bottom part of $x^2+y^2=4$. Calculate $\int_C y\,ds$



$$C: \quad \vec{r}(t) = \langle \overset{x}{2\cos t}, \overset{y}{2\sin t}\rangle \qquad \pi \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$$

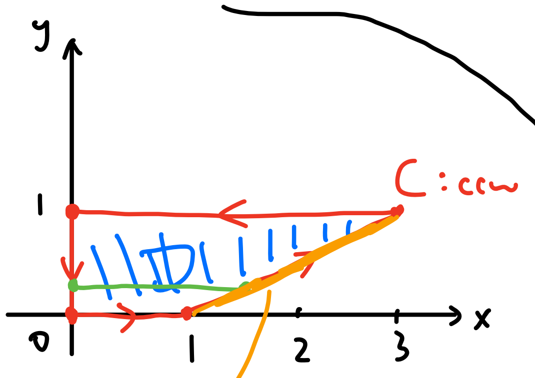
$$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

$$\int_C y\,ds = \int_\pi^{2\pi} 2\sin t \cdot 2\,dt = 4\cdot(-\cos t)\Big|_\pi^{2\pi}$$

$$= 4\cdot(-1-1) = -8$$

Ex C is the closed curve connecting $(0,0)$, $(1,0)$, $(3,1)$, $(0,1)$ in order, by segments. Calculate

$$\oint_C (x^3 + \overset{P}{y})\,dx + (\overset{Q}{-xy - \sin(y^2)})\,dy$$ by Green's Theorem



$y = \frac{1}{2}x - \frac{1}{2}$

$\frac{1}{2}x = y + \frac{1}{2}$

$x = 2y + 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y - 1$$

Green's

$$= \iint_D (-y-1)\,dA$$

$$= \int_0^1 \int_0^{2y+1} (-y-1)\,dx\,dy$$

$$= \int_0^1 \underbrace{(-y-1)(2y+1)}_{-2y^2-3y-1}\,dy$$

$$= \left(-\frac{2}{3}y^3 - \frac{3}{2}y^2 - y\right)\Big|_0^1$$

$$= -\frac{2}{3} - \frac{3}{2} - 1 = -\frac{19}{6}.$$

Ex $f(x,y) = x^2y^2 - 2y + 1$ $\qquad f(1,2) = 1^2 \cdot 2^2 - 2\cdot 2 + 1 = 1$

① ∇f

② direction of fastest decrease, at $(1,2)$, expressed in unit vector

③ directional derivative in $\vec{u} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$, at $(1,2)$

④ linear approximation at $(1,2)$. use it to approximate $f(1.01, 1.97)$

① $\nabla f = \langle 2xy^2, 2x^2y - 2 \rangle$

② $\nabla f(1,2) = \langle 2\cdot 1\cdot 2^2, 2\cdot 1^2\cdot 2 - 2 \rangle = \langle 8, 2 \rangle$

direction of fastest decrease : $-\nabla f(1,2) = \langle -8, -2 \rangle$

length $= \sqrt{64+4} = \sqrt{68}$

normalize

$\frac{1}{\sqrt{68}} \langle -8, -2 \rangle$

③ $D_{\vec{u}} f(1,2) = \vec{u} \cdot \nabla f(1,2) = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle \cdot \langle 8, 2 \rangle$

$= 4\sqrt{2} + (-\sqrt{2}) = 3\sqrt{2}$

④ $L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$= 1 + 8(x-1) + 2(y-2)$

$f(1.01, 1.97) \approx L(1.01, 1.97) = 1 + 8(1.01-1) + 2(1.97-2)$

$= 1 + 0.08 - 0.06 = 1.02$

Ex $f(x,y) = x^3 + x^2 y - y^2$

Find all critical pts. Determine type by second derivative test.

$$\begin{cases} f_x = 3x^2 + 2xy = 0 \\ f_y = x^2 - 2y = 0 \end{cases} \rightsquigarrow y = \frac{x^2}{2}$$

$3x^2 + 2x \cdot \frac{x^2}{2} = 0$

$3x^2 + x^3 = 0$

$x^2(3+x) = 0$

$x = -3$ or $x = 0$

$\Downarrow$ $\quad$ $\Downarrow$

$y = \frac{9}{2}$ $\quad$ $y = 0$

$\Rightarrow$ crit pts : $(-3, \frac{9}{2})$, $(0,0)$.

$f_{xx} = 6x + 2y \qquad f_{xy} = 2x \qquad f_{yy} = -2.$

At $(-3, \frac{9}{2})$: $\quad f_{xx} = 6\cdot(-3) + 2\cdot\frac{9}{2} = -9$

$f_{xy} = 2\cdot(-3) = -6 \qquad f_{yy} = -2$

$$D = f_{xx} f_{yy} - f_{xy}^2 = (-9)(-2) - (-6)^2 = -18 < 0$$

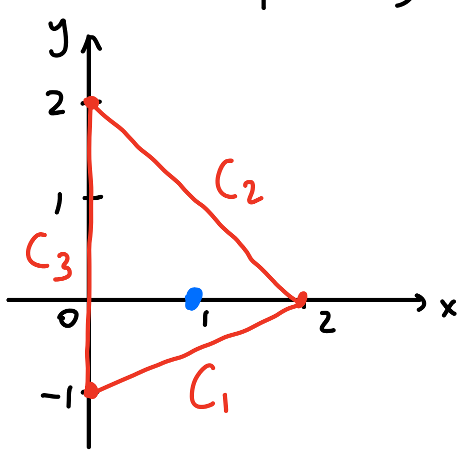
neither max nor min.

At $(0,0)$: $\quad f_{xx} = 0 \qquad f_{xy} = 0 \qquad f_{yy} = -2$

$$D = f_{xx} f_{yy} - f_{xy}^2 = 0\cdot(-2) - 0^2 = 0 \quad \text{inconclusive}$$

$\underline{\text{Ex}}$ Find global max/min of $f(x,y) = x^2 - y^2 - 2x$

on the following triangular region:



① crit pts in domain.

$$\begin{cases} f_x = 2x - 2 = 0 & x = 1 \\ f_y = -2y = 0 & y = 0 \end{cases}$$

$(1,0)$ in domain

$f(1,0) = 1^2 - 0^2 - 2\cdot 1 = \underline{-1}$

② candidates on boundary.

C_1: $\quad y = \frac{1}{2}x - 1, \quad 0 \le x \le 2$

$$f = x^2 - (\tfrac{1}{2}x - 1)^2 - 2x = x^2 - (\tfrac{1}{4}x^2 - x + 1) - 2x$$

$$= \frac{3}{4}x^2 - x - 1$$

$\Big\} \frac{d}{dx}$

$\frac{3}{2}x - 1 = 0 \qquad x = \frac{2}{3}$

At $x = 0$, $f = \underline{-1}$

At $x = 2$, $f = \frac{3}{4}\cdot 4 - 2 - 1 = \underline{0}$

At $x = \frac{2}{3}$, $f = \frac{3}{4}\cdot\frac{4}{9} - \frac{2}{3} - 1 = \underline{-\frac{4}{3}}$

$C_2: \ y = -x+2, \quad 0 \le x \le 2$

$f = x^2 - (-x+2)^2 - 2x = x^2 - (x^2 - 4x + 4) - 2x$

$= 2x - 4$

At $x=0$, $f = \underset{\sim}{-4}$

At $x=2$, $f = \underset{\sim}{0}$

$C_3: \ x=0, \quad -1 \le y \le 2$

$f = -y^2$

At $y=0$, $f = \underset{\sim}{0}$

At $y=2$, $f = \underset{\sim}{-4}$

③ Conclude:

global max $= 0$, achieved at $(2,0), (0,0)$

global min $= -4$, achieved at $(0,2)$

Ex Distance from $P(3,-1,4)$ to the plane $2x - y + 4z = 1$

$\vec{n} = \langle 2, -1, 4 \rangle$

$Q(\frac{1}{2}, 0, 0)$

$\vec{QP} = \langle \frac{5}{2}, -1, 4 \rangle$

$$\frac{|\vec{QP} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|5+1+16|}{\sqrt{4+1+16}} = \frac{22}{\sqrt{21}}$$

Ex Eq. of plane containing $P(0,1,2)$, $Q(1,2,3)$, $R(2,3,-1)$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$\vec{PQ} = \langle 1,1,1 \rangle$

$\vec{PR} = \langle 2,2,-3 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 2 & -3 \end{vmatrix} = \langle -5, 5, 0 \rangle$$

$\Rightarrow -5(x-0) + 5(y-1) + 0 \cdot (z-2) = 0$

<u>Ex</u> Distance from $P(-1,2,0)$ to the line

$$\vec{r}(t) = \langle 1+t, 2-t, 3t \rangle$$

$$\frac{\| \vec{PM} \times \vec{v} \|}{\| \vec{v} \|}$$

$$= \frac{\sqrt{0+36+4}}{\sqrt{1+1+9}} = \boxed{\frac{\sqrt{40}}{\sqrt{11}}}$$

$\vec{v} = \langle 1, -1, 3 \rangle$

$M(1,2,0)$

$\vec{PM} = \langle 2, 0, 0 \rangle$

$$\vec{PM} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= \langle 0, -6, -2 \rangle$$