

Section 5.6:

Mass:

For a solid Q with density $\rho(x, y, z)$:

$$m = \iiint_Q \rho(x, y, z) \, dV$$

For a lamina R with density $\rho(x, y)$:

$$m = \iint_R \rho(x, y) \, dA$$

Center of mass:

For a solid Q with density $\rho(x, y, z)$: it is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\iiint_Q x\rho(x, y, z) \, dV}{\iiint_Q \rho(x, y, z) \, dV}, \quad \bar{y} = \frac{\iiint_Q y\rho(x, y, z) \, dV}{\iiint_Q \rho(x, y, z) \, dV}, \quad \bar{z} = \frac{\iiint_Q z\rho(x, y, z) \, dV}{\iiint_Q \rho(x, y, z) \, dV}$$

For a lamina R with density $\rho(x, y)$: it is (\bar{x}, \bar{y}) where

$$\bar{x} = \frac{\iint_R x\rho(x, y) \, dA}{\iint_R \rho(x, y) \, dA}, \quad \bar{y} = \frac{\iint_R y\rho(x, y) \, dA}{\iint_R \rho(x, y) \, dA}$$

Moments of inertia:

For a solid Q with density $\rho(x, y, z)$:

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z) \, dV, \quad I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z) \, dV$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z) \, dV$$

For a lamina R with density $\rho(x, y)$:

$$I_x = \iint_R y^2\rho(x, y) \, dA, \quad I_y = \iint_R x^2\rho(x, y) \, dA$$

$$I_0 = \iint_R (x^2 + y^2)\rho(x, y) \, dA = I_x + I_y$$

Total time: 80 minutes.

Total points: 100.

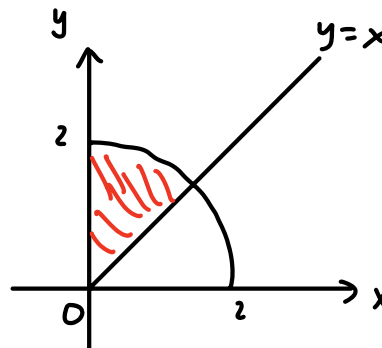
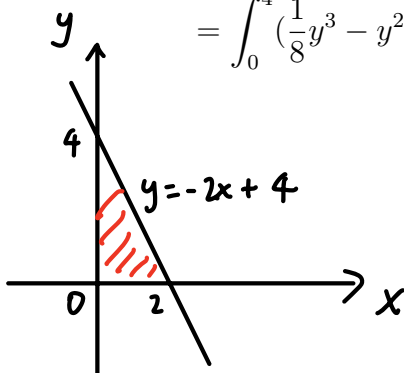
Problem 1 (15 + 15 = 30 points). Calculate double integrals:

(1) $\iint_D xy \, dA$, where D is the region bounded by the x -axis, the y -axis, and the line $y = -2x + 4$.

$$\begin{aligned} \int_0^2 \int_0^{-2x+4} xy \, dy \, dx &= \int_0^2 x \cdot \frac{1}{2} y^2 \Big|_{y=0}^{-2x+4} \, dx = \int_0^2 x \cdot \frac{1}{2} (-2x+4)^2 \, dx \\ &= \int_0^2 (2x^3 - 8x^2 + 8x) \, dx = \left(\frac{1}{2} x^4 - \frac{8}{3} x^3 + 4x^2 \right) \Big|_{x=0}^2 = \frac{1}{2} 2^4 - \frac{8}{3} 2^3 + 4 \cdot 2^2 = \frac{8}{3} \end{aligned}$$

Another way:

$$\begin{aligned} \int_0^4 \int_0^{-\frac{1}{2}y+2} xy \, dx \, dy &= \int_0^4 \frac{1}{2} x^2 y \Big|_{x=0}^{-\frac{1}{2}y+2} \, dy = \int_0^4 \frac{1}{2} \left(-\frac{1}{2}y+2\right)^2 y \, dy \\ &= \int_0^4 \left(\frac{1}{8}y^3 - y^2 + 2y\right) \, dy = \left(\frac{1}{32}y^4 - \frac{1}{3}y^3 + y^2\right) \Big|_{y=0}^4 = \frac{1}{32} 4^4 - \frac{1}{3} 4^3 + 4^2 = \frac{8}{3} \end{aligned}$$



(2) $\iint_D x \, dA$, where D is the region inside the circle $x^2 + y^2 = 4$, in the first quadrant, above the line $y = x$.

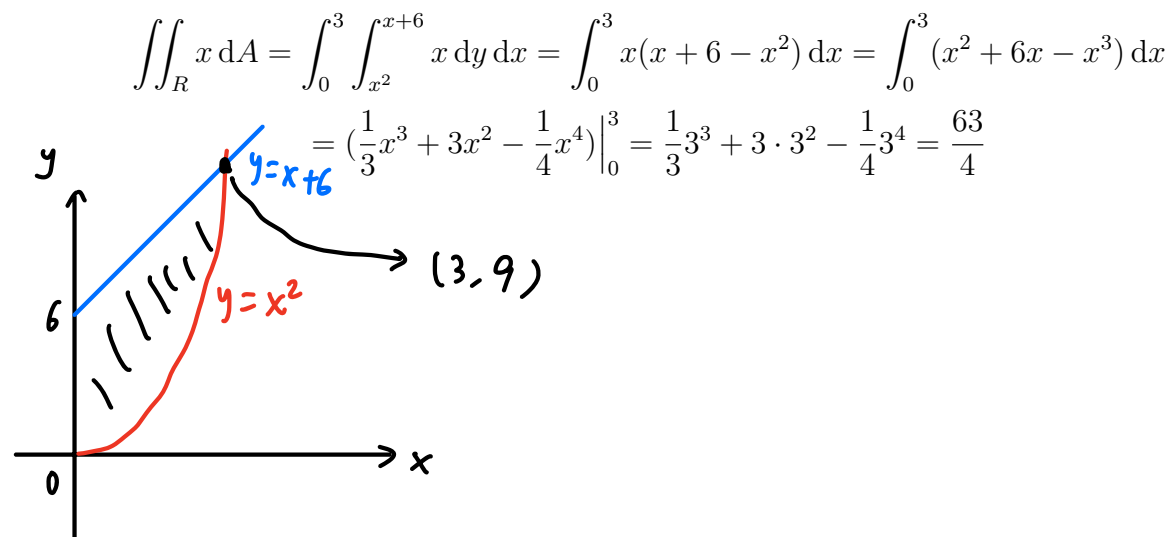
By polar coordinates,

$$\int_0^2 \int_{\pi/4}^{\pi/2} r \cos \theta \cdot r \, d\theta \, dr = \int_0^2 r^2 \, dr \cdot \int_{\pi/4}^{\pi/2} \cos \theta \, d\theta = \frac{1}{3} r^3 \Big|_0^2 \cdot \sin \theta \Big|_{\pi/4}^{\pi/2} = \frac{8}{3} \left(1 - \frac{\sqrt{2}}{2}\right)$$

Problem 2 (15 + 10 = 25 points). A lamina R is inside the first quadrant, bounded by the parabola $y = x^2$ and the line $y = x + 6$, whose density is given by $\rho(x, y) = x$.

(1) Calculate its mass.

We first calculate the intersection points of $y = x^2$ and $y = x + 6$. Setting the two equations together, we get $x^2 - x - 6 = 0$. $(x - 3)(x + 2) = 0$. Therefore the intersection point in the first quadrant is $(3, 9)$.



(2) Calculate the x -coordinate of its center of mass.

$$\iint_R x \cdot x \, dA = \int_0^3 \int_{x^2}^{x+6} x^2 \, dy \, dx = \int_0^3 x^2(x+6-x^2) \, dx = \int_0^3 (x^3 + 6x^2 - x^4) \, dx$$

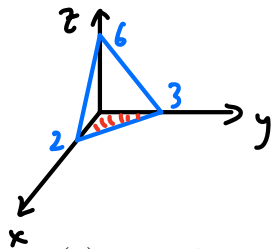
$$= \left(\frac{1}{4}x^4 + 2x^3 - \frac{1}{5}x^5 \right) \Big|_0^3 = \frac{1}{4}3^4 + 2 \cdot 3^3 - \frac{1}{5}3^5 = \frac{19 \cdot 3^3}{20}$$

$$\bar{x} = \frac{\frac{19 \cdot 3^3}{20}}{\frac{63}{4}} = \frac{57}{35}$$

The numbers in this problem are complicated, and simplification is not required. For example, an answer $\bar{x} = \frac{\frac{1}{4}3^4 + 2 \cdot 3^3 - \frac{1}{5}3^5}{\frac{1}{3}3^3 + 3 \cdot 3^2 - \frac{1}{4}3^4}$ is allowed.

Problem 3 ($5 \times 3 = 15$ points). Express the triple integral $\iiint_E e^x dV$ as an iterated integral, for the following regions E . You DO NOT need to calculate this iterated integral.

(1) E is the region in the first octant, below the plane $3x + 2y + z = 6$.



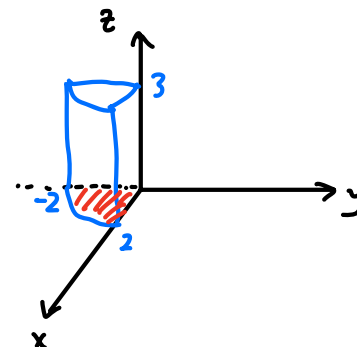
$$\int_0^2 \int_0^{(6-3x)/2} \int_0^{6-3x-2y} e^x dz dy dx$$

(2) E is the region in the cylinder $x^2 + y^2 = 4$, with $x \geq 0$, $y \leq 0$, $z \geq 0$, below the plane $z = 3$.

$$\int_0^2 \int_{3\pi/2}^{2\pi} \int_0^3 e^{r \cos \theta} dz r d\theta dr$$

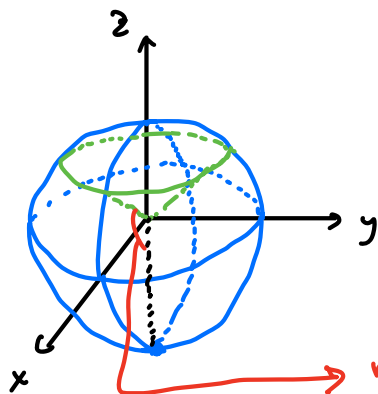
or

$$\int_0^2 \int_{-\sqrt{4-x^2}}^0 \int_0^3 e^x dz dy dx$$



(3) E is the region in the sphere $x^2 + y^2 + z^2 = 9$, below the cone $z = \sqrt{x^2 + y^2}$.

$$\int_0^3 \int_{\pi/4}^{\pi} \int_0^{2\pi} e^{\rho \sin \varphi \cos \theta} \rho^2 \sin \varphi d\theta d\varphi d\rho$$



range of φ : $\frac{\pi}{4} \rightsquigarrow \pi$

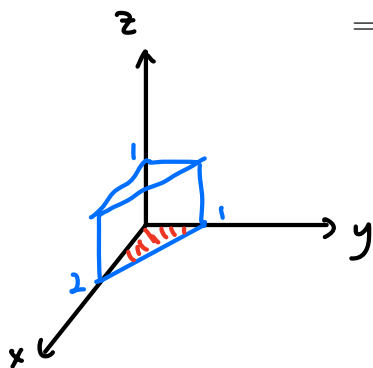
Problem 4 (15 points). Let E be the region in the first octant, bounded by the planes $x + 2y = 2$ and $z = 1$. Calculate

$$\iiint_E yz \, dV$$

$$\begin{aligned} \int_0^1 \int_0^{2-2y} \int_0^1 yz \, dz \, dx \, dy &= \int_0^1 \int_0^{2-2y} y \cdot \frac{1}{2} z^2 \Big|_{z=0}^1 \, dx \, dy = \frac{1}{2} \int_0^1 \int_0^{2-2y} y \, dx \, dy \\ &= \frac{1}{2} \int_0^1 y(2-2y) \, dy = \int_0^1 (y-y^2) \, dy = \left(\frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \Big|_{y=0}^1 = \frac{1}{6} \end{aligned}$$

Another way:

$$\begin{aligned} \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^1 yz \, dz \, dy \, dx &= \int_0^2 \int_0^{1-\frac{x}{2}} y \cdot \frac{1}{2} z^2 \Big|_{z=0}^1 \, dy \, dx = \frac{1}{2} \int_0^2 \int_0^{1-\frac{x}{2}} y \, dy \, dx \\ &= \frac{1}{2} \int_0^2 \frac{1}{2} y^2 \Big|_{y=0}^{1-\frac{x}{2}} \, dx = \frac{1}{4} \int_0^2 \left(1 - \frac{x}{2}\right)^2 \, dx = \frac{1}{4} \int_0^2 \left(1 - x + \frac{1}{4} x^2\right) \, dx \\ &= \frac{1}{4} \left(x - \frac{1}{2} x^2 + \frac{1}{12} x^3 \right) \Big|_{x=0}^2 = \frac{1}{4} \left(2 - \frac{1}{2} 2^2 + \frac{1}{12} 2^3 \right) = \frac{1}{6} \end{aligned}$$



Problem 5 (15 points). Let Q be the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, with density $\rho(x, y, z) = 3$. Calculate its moment of inertia about the z -axis.

By spherical coordinates,

$$\begin{aligned}
 I_z &= \iiint_Q (x^2 + y^2) \cdot 3 \, dV = 3 \int_1^2 \int_0^\pi \int_0^{2\pi} (\rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta) \rho^2 \sin \varphi \, d\theta \, d\varphi \, d\rho \\
 &= 3 \int_1^2 \int_0^\pi \int_0^{2\pi} \rho^2 \sin^2 \varphi \cdot \rho^2 \sin \varphi \, d\theta \, d\varphi \, d\rho \\
 &= 3 \int_1^2 \rho^4 \, d\rho \cdot \int_0^\pi \sin^3 \varphi \, d\varphi \cdot \int_0^{2\pi} 1 \, d\theta \\
 &= 3 \cdot \frac{1}{5} \rho^5 \Big|_1^2 \cdot \left(-\cos \varphi + \frac{1}{3} \cos^3 \varphi \right) \Big|_0^\pi \cdot 2\pi = 3 \cdot \frac{31}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{248}{5} \pi
 \end{aligned}$$

Here we calculated the anti-derivative

$$\int \sin^3 \varphi \, d\varphi = \int (1 - \cos^2 \varphi) \sin \varphi \, d\varphi = - \int (1 - u^2) \, du = -\left(u - \frac{1}{3} u^3\right) + C = -\cos \varphi + \frac{1}{3} \cos^3 \varphi + C$$

using the substitution $u = \cos \varphi$.