Math 2500 Quiz 9 Solution (20260415)
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Total time: 10 minutes.

Green's Theorem (circulation form): Let C be a simple closed curve in 2D, traveling counterclockwise, enclosing the region D. Let $\mathbf{F} = \langle P, Q \rangle$. Then

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$

Problem 1 (6 points). Let C be the boundary of the rectangle $[0, 1] \times [0, 2]$, traveling clockwise. Use Green's Theorem to calculate:

$$\int_C \langle e^x \sin y + y^2, e^x \cos y \rangle \cdot \mathrm{d}\mathbf{r}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^x \cos y - (e^x \cos y + 2y) = -2y$$

Applying Green's Theorem, there is an extra negative sign because C is clockwise:

$$\int_C \langle e^x \sin y + y^2, e^x \cos y \rangle \cdot \mathrm{d}\mathbf{r} = \iint_D 2y \,\mathrm{d}A = \int_0^1 \int_0^2 2y \,\mathrm{d}y \,\mathrm{d}x$$

$$= \int_0^1 1 \,\mathrm{d}x \cdot \int_0^2 2y \,\mathrm{d}y = 1 \cdot y^2 \Big|_0^2 = 4$$

Problem 2 (4 points). Calculate divergence of the following vector fields:

$$\mathbf{F}(x, y) = \langle e^{xy}, y^2 \sin x \rangle, \quad \mathbf{G}(x, y, z) = \langle x^2 + yz,\ xz - 2yz,\ x^2 y + x^3 z^2 \rangle$$

$$\nabla \cdot \mathbf{F} = y e^{xy} + 2y \sin x$$

$$\nabla \cdot \mathbf{G} = 2x - 2z + 2x^3 z$$